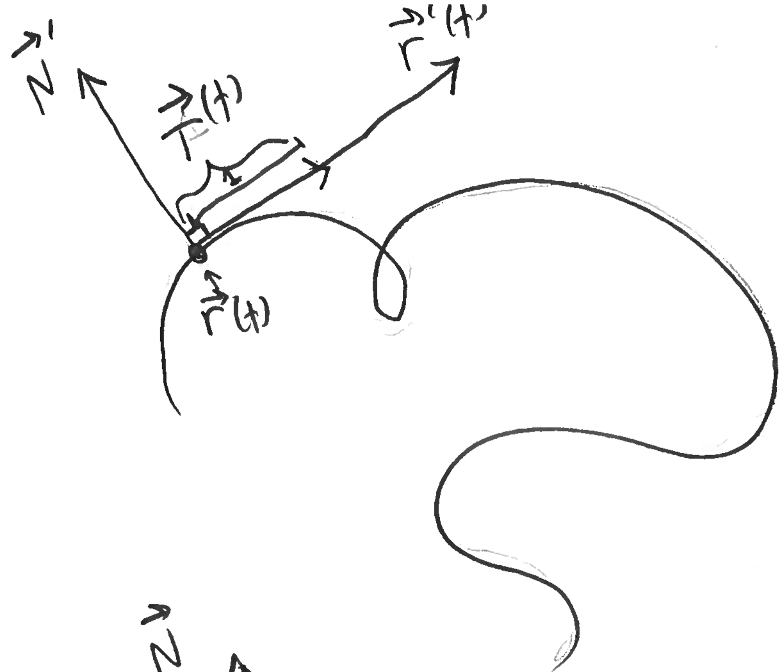
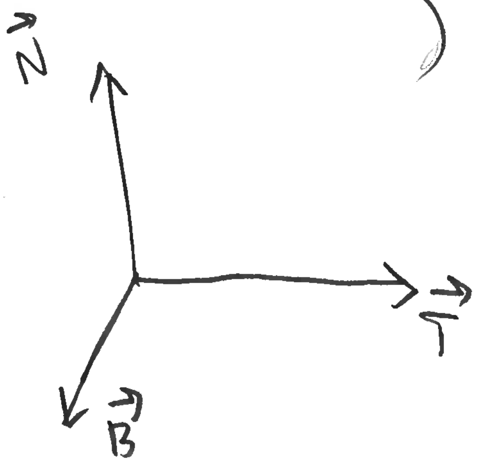


$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$



$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Ex: Let $\vec{r}(t) = \langle 2\sin(t), t, 1 \rangle$

Find $\vec{T}, \vec{N}, \vec{B}$

Soln: $\vec{r}'(t) = \langle 2\cos(t), 1, 0 \rangle$

$$\|\vec{r}'(t)\| = \sqrt{4\cos^2(t) + 1}$$

$$\vec{T}(t) = \left\langle \frac{2\cos(t)}{\sqrt{4\cos^2(t)+1}}, \frac{1}{\sqrt{4\cos^2(t)+1}}, 0 \right\rangle$$

$$\frac{d}{dt}\left[\frac{2\cos(t)}{(4\cos^2(t)+1)^{1/2}}\right] = \frac{\left(\sqrt{4\cos^2(t)+1}\right)(-2\sin(t)) - \frac{2\cos(t)\,[8\cos(t)(-\sin(t))]}{2(4\cos^2(t)+1)^{1/2}}}{4\cos^2(t)+1}$$

Ex: $\vec{r}(t) = \langle 2\cos(5t), 5, 2\sin(5t) \rangle$

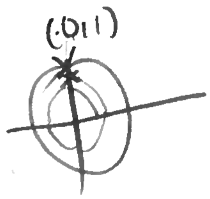
$$\vec{r}\,'(t) = \langle -10\sin(5t), 0, 10\cos(5t) \rangle$$

$$\vec{T}(t) = \frac{\langle -10\sin(5t), 0, 10\cos(5t) \rangle}{\sqrt{100\sin^2(5t) + 100\cos^2(5t)}} \leftarrow \sqrt{100} = 10$$

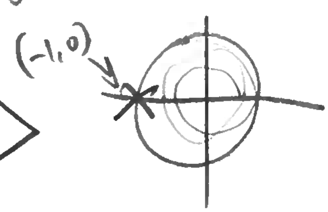
$$= \langle -\sin(5t), 0, \cos(5t) \rangle$$

$$\vec{T}\,'(t) = \langle -5\cos(5t), 0, -5\sin(5t) \rangle$$

$$\vec{N}(t) = \frac{\langle -5\cos(5t), 0, -5\sin(5t) \rangle}{\sqrt{25\cos^2(5t) + 25\sin^2(5t)}}$$



$$= \langle -\cos(5t), 0, -\sin(5t) \rangle$$



$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$= \langle -\sin(5t), 0, \cos(5t) \rangle \times \langle -\cos(5t), 0, -\sin(5t) \rangle$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(5t) & 0 & \cos(5t) \\ -\cos(5t) & 0 & -\sin(5t) \end{bmatrix}$$

$$= \langle 0, -(\sin^2(5t) - (-\cos^2(5t))), 0 \rangle$$

$$= \langle 0, -1, 0 \rangle$$

$t = \frac{\pi}{2}$

$\vec{r}(\frac{\pi}{2}) = \langle 0, 5, 2 \rangle$

$\vec{T}(\frac{\pi}{2}) = \langle -1, 0, 0 \rangle$

$\vec{N}(\frac{\pi}{2}) = \langle 0, 0, -1 \rangle$

$\vec{B}(\frac{\pi}{2}) = \langle 0, -1, 0 \rangle$

at $t = \pi$

$\vec{T}(t) = \langle 0, 0, -1 \rangle$

$\vec{N}(t) = \langle 1, 0, 0 \rangle$

$\vec{B}(t) = \langle 0, -1, 0 \rangle$

③

FTC

$\int f'(t)\,dt = f(t) + C$

$\int_a^b f'(t)\,dt = f(t)\Big|_a^b$

Ex: Solve the ODEq (ordinary diff eq)

$$\begin{cases} \vec{r}\,'(t) = \langle \sin(4t), \sin(3t), 7t \rangle \\ \vec{r}(0) = \langle 7, 4, 7 \rangle \end{cases}$$

determines the "+C"

Find $\vec{r}(t)$.

Soln: By FTC:

$\int \sin(3t)\,dt = \frac{1}{3}\int \sin(u)\,du$
$= 1$

$u = 3t$

$\frac{1}{3}du = dt$

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt$$

$$= \left\langle \int \sin(4t)\,dt, \int \sin(3t)\,dt, \int 7t\,dt \right\rangle$$

$$= \left\langle \frac{-1}{4}\cos(4t) + C_1, \frac{-1}{3}\cos(3t) + C_2, \frac{7}{2}t^2 + C_3 \right\rangle$$

$$= \left\langle -\frac{1}{4}\cos(4t), -\frac{1}{3}\cos(3t), \frac{7}{2}t^2 \right\rangle + \vec{C}$$

Now,

$$\langle 7, 4, 7 \rangle = \vec{r}(0) = \left\langle -\frac{1}{4}, -\frac{1}{3}, 0 \right\rangle + \vec{C}$$

given

$$\boxed{\begin{aligned} \vec{C} &= \langle 7, 4, 7 \rangle - \left\langle -\frac{1}{4}, -\frac{1}{3}, 0 \right\rangle \\ &= \left\langle \frac{29}{4}, \frac{13}{3}, 7 \right\rangle \end{aligned}}$$

$+, -, \div, \times$ introduce variable numbers

$$
\begin{tikzcd}
\text{Arithmetic} \arrow[r] \arrow[d] & \text{Algebra} \arrow[d] \\
\text{Calculus} \arrow[r] & \text{Diff Eq}
\end{tikzcd}
$$

$\frac{d}{dx}$ $\int \cdot\, dx$ introduce variable functions

$$y'' = -y$$

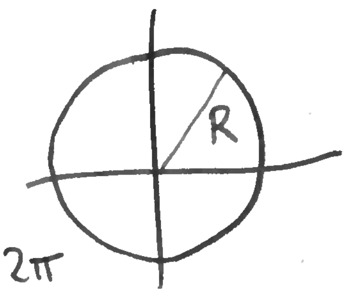
$$
\begin{array}{ll}
\sin(t) & \cos(t) \\
\downarrow d/dt & \\
\cos(t) & \\
\downarrow d/dt & \\
-\sin(t) &
\end{array}
$$

<u>Arc length</u> – length of the space curve

if $\vec{r}(t)$, $a \le t \le b$ then its length is

$$L = \int_a^b \|\vec{r}\,'(t)\|\,dt$$

<u>EX</u>: Find arclength of $\begin{cases} \vec{r}(t) = \langle R\cos(t), R\sin(t)\rangle \\ 0 \le t \le 2\pi \end{cases}$



$$L = \int_0^{2\pi} \|\langle -R\sin(t), R\cos(t)\rangle\|\,dt$$

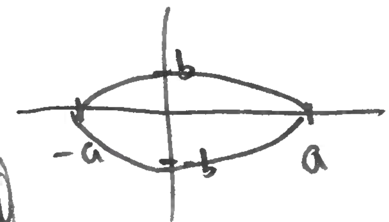
$$= \int_0^{2\pi} \sqrt{R^2\sin^2(t) + R^2\cos^2(t)}\,dt$$

$$= R\int_0^{2\pi} 1\,dt = 2\pi R$$

Ex: Find length of $\begin{cases} \vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle \\ 0 \le t \le 2\pi \end{cases}$

$a > 0 \quad b > 0$

Soln: ~~Assume $a^2 < b^2$.~~



$$L = \int_0^{2\pi} \| \langle -a\sin(t), b\cos(t)\rangle \| \, dt$$

$$= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + b^2\cos^2(t)} \, dt$$

$\sin^2 t + \cos^2 t = 1$

$\sin^2(t) = 1 - \cos^2(t)$

$$= a\int_0^{2\pi} \sqrt{\sin^2(t) + \frac{b^2}{a^2}\cos^2(t)} \, dt$$

$$= a\int_0^{2\pi} \sqrt{1 + \left(\frac{b^2}{a^2} - 1\right)\cos^2(t)} \, dt$$

elliptic integral of 2nd kind

$$= b\int_0^{2\pi} \sqrt{\frac{a^2}{b^2}\sin^2(t) + \cos^2(t)} \, dt$$

$=$