Reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

$\boxed{x=\frac{1}{2}}$ $\Downarrow$

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} \quad = 1$$

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \pi$$

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

Ex: Recall $(If)(x) = \int_0^x f(t)\,dt$

Let $f(x) = x^4$.

$$If = \int_0^x t^4\,dt = \frac{t^5}{5}\Big|_0^x = \frac{x^5}{5}$$

$$I^2 f = I(If) = I\left(\frac{x^5}{5}\right) = \int_0^x \frac{t^5}{5}\,dt = \frac{1}{5}\,\frac{t^6}{6}\Big|_0^x = \frac{x^6}{5\cdot 6}$$

$$I^3 f = I(I^2 f) = I\left(\frac{x^6}{5\cdot 6}\right) = \frac{1}{5\cdot 6}\int_0^x t^6\,dt = \frac{x^7}{5\cdot 6\cdot 7}$$

$\vdots$

$$\boxed{I^n f = \frac{x^{n+4}}{(4+1)(4+2)\cdots(4+n)}}$$

General pattern: if $f(x) = x^k$, then

$$I^n f = \frac{x^{k+n}}{(k+1)(k+2)\cdots(k+n)}$$

Mathematician Cauchy (1789-1857)

Theorem: $(I^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt$

$\Gamma(n)$

weird — usually don't allow the variable of a function inside

Ex: $I^2 x^4 = \frac{1}{(2-1)!}\int_0^x (x-t)^{2-1} t^4\,dt$

$= \left(\frac{1}{1!}\right)\int_0^x xt^4 - t^5\,dt$

$= \int_0^x x t^4\,dt - \int_0^x t^5\,dt$

$= x\int_0^x t^4\,dt - \int_0^x t^5\,dt$

$= x\left[\frac{t^5}{5}\Big|_0^x - \frac{t^6}{6}\Big|_0^x = x\left(\frac{x^5}{5} - 0\right) - \left(\frac{x^6}{6} - 0\right)\right.$

$= \frac{x^6}{5} - \frac{x^6}{6} = \frac{6x^6 - 5x^6}{5\cdot 6} = \frac{x^6}{5\cdot 6}$

From Cauchy's thm, we can give meaning to, say, a "$\frac{1}{2}$ integral" by setting $n = \frac{1}{2}$:

$$\left(I^{1/2} f\right)(x) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^x (x-t)^{\frac{1}{2}-1} f(t)\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_0^x \frac{f(t)}{\sqrt{x-t}}\, dt$$

What is $\frac{1}{2}$ integral of $f(t) = 1$?

$$\left(I^{1/2} 1\right)(x) = \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-t}}\, dt$$

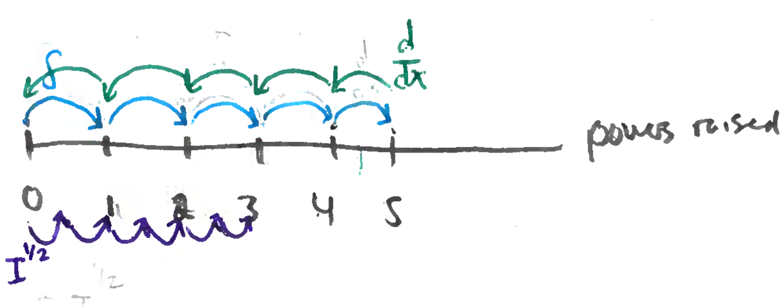
$$\left(\text{reality: } \left(I^{1/2} 1\right)(x) = x^{0+1/2}\right.$$

$$\int t = \frac{t^2}{2}$$

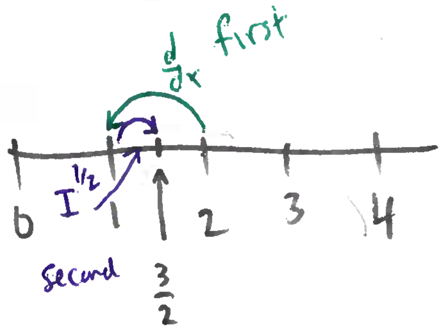
$$I^{1/2} t = (\text{const})\, t^{1+1/2}$$

General: fractional integral of order α

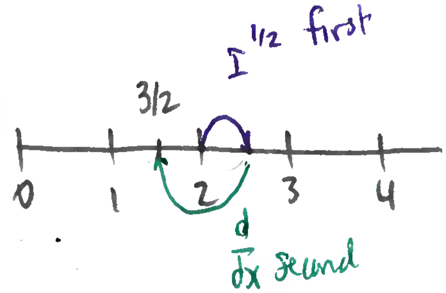
$$\left(I^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$



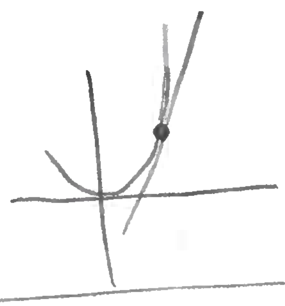
Ex: $\frac{1}{2}$ deriv of x^2



Caputo fractional derivative



Riemann-Liouville fract. deriv.



$$\Delta f(t) = f(t+1) - f(t)$$

$$\Delta(fg) = (\Delta f(t))g(t) + \underbrace{f(t+1)}_{f^\sigma}\Delta g(t)$$

$$\Delta\left(\frac{f}{g}\right) = \frac{g(\Delta f) - f(\Delta g)}{g g^\sigma}$$