Gamma function

Γ ← capitel Greek letter gamma

Def: $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt$

$= \lim_{w\to\infty} \int_0^{w} t^{x-1} e^{-t}\,dt$

$D^{-1}(e^{-t}) = -e^{-t} + C$

$\Updownarrow$

$\frac{d}{dt}(-e^{-t}) = e^{-t}$

$\| $

$-\frac{d}{dt} e^{-t}$

$-\underbrace{\frac{d(-t)}{dt}}_{=-1} \frac{d}{d(-t)} e^{-t}$

Compute

$\Gamma(1) \overset{def}{=} \lim_{w\to\infty} \int_0^{w} t^{1-1} e^{-t}\,dt$

$t^0 = 1$

$= \lim_{w\to\infty} \int_0^{w} e^{-t}\,dt$

$= \lim_{w\to\infty} \left[-e^{-t} + C\right|_0^{w}$

$= \lim_{w\to\infty} \left[(-e^{-w} + \cancel{C}) - (-\overset{1}{e^0} + \cancel{C})\right]$

$= \lim_{w\to\infty} (1 - \underset{\to 0}{e^{-w}})$

$= 1$

Compute

$$\Gamma(2) = \lim_{w\to\infty} \int_0^w t^{2-1} e^{-t}\,dt$$

$$= \lim_{w\to\infty} \int_0^w te^{-t}\,dt \leftarrow$$

look!

$$\frac{d}{dt}\left(-e^{-t}(t+1)\right)$$

$$= -\left[-e^{-t}(t+1) + e^{-t}(1)\right]$$

$$= -\left[-te^{-t} - \cancel{e^{-t}} + \cancel{e^{-t}}\right]$$

$$= te^{-t}$$

$$\updownarrow$$

$$D^{-1}\left(te^{-t}\right) = -e^{-t}(t+1) + C$$

$$= \lim_{w\to\infty} \left[-e^{-t}(t+1) + C\right]_0^w$$

$$= -\lim_{w\to\infty} \left[\left(e^{-w}(w+1) + \cancel{C}\right) - \left(e^{0}(0+1) + \cancel{C}\right)\right]$$

$$= -\lim_{w\to\infty} \left[\underbrace{\frac{w+1}{e^{w}}}_{0} - 1\right]$$

$$= \underline{1}$$

Similarly:

$$\Gamma(3) = 2 = 2!$$

$$\Gamma(4) = 6 = 3!$$

$$\Gamma(5) = 24 = 4!$$

$$\vdots$$

Fact: $\frac{d}{dt}\left(t^{x}e^{-t}\right) = xt^{x-1}e^{-t} - t^{x}e^{-t}$

$$\Big\downarrow \int_0^w$$

$$\int_0^w \boxed{\frac{d}{dt}\left(t^{x}e^{-t}\right)}dt = \int_0^w xt^{x-1}e^{-t} - t^{x}e^{-t}\,dt$$

$$\int_0^w \frac{d}{dt}\left(t^{x}e^{-t}\right)dt = t^{x}e^{-t}\Big|_0^w = w^{x}e^{-w} - 0$$

$$\int_0^w xt^{x-1}e^{-t} - t^{x}e^{-t}\,dt = x\int_0^w t^{x-1}e^{-t}dt - \int_0^w t^{x}e^{-t}dt$$

$$w^{x}e^{-w} = x\int_0^w t^{x-1}e^{-t}dt - \int_0^w t^{x}e^{-t}dt$$

$$\Big\downarrow \text{take } \lim_{w\to\infty}$$

$$0 = x\Gamma(x) - \Gamma(x+1)$$

$$(\star)\quad \boxed{\Gamma(x+1) = x\Gamma(x)}$$

$x=2$: $\boxed{\Gamma(3) = 2}\,\underbrace{\Gamma(2)}_{=1}$, $\Gamma(3) = 2!$

$x=3$: $\Gamma(4) = 3\Gamma(3)$, $\Gamma(3) = 2$, $\Gamma(4) = 6 = 3!$

$x=4$: $\Gamma(5) = 4\Gamma(4) = 4!$

$$\boxed{\Gamma(n+1) = n!}$$

$$x\Gamma(x) = \Gamma(x+1)$$

$$x = \frac{1}{2} \rightarrow \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \underbrace{\Gamma\left(\frac{1}{2}+1\right)}_{``="\left(\frac{1}{2}\right)!}$$

TURNS OUT: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$``\left(\frac{1}{2}\right)!" = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

What is $\left(\frac{3}{2}\right)!$? $= \Gamma\left(\frac{3}{2}\right)$

Set $x = \frac{3}{2}$ in $x\Gamma(x) = \Gamma(x+1)$

$$\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \underbrace{\Gamma\left(\frac{3}{2}+1\right)}_{=``\left(\frac{3}{2}\right)!"}$$

$$\frac{3}{2}\frac{\sqrt{\pi}}{2} = ``\left(\frac{3}{2}\right)!"$$

$$\left(\frac{3}{2}\right)! = \frac{3\sqrt{\pi}}{4}$$

Consider operator I

Def: $If = \int_0^x f(t)\,dt$

Ex: $f(x) = x^2$

$$If = \int_0^x t^2\,dt = \left.\frac{t^3}{3}\right|_0^x = \frac{x^3}{3}$$

$$I^2 f = I(If) = I\left(\frac{x^3}{3}\right) = \int_0^x \frac{t^3}{3}\,dt$$

$$= \left.\frac{1}{3}\frac{t^4}{4}\right|_0^x$$

$$= \frac{x^4}{3 \cdot 4}$$

$$I^3 f = I(I^2 f) = I\left(\frac{x^4}{3 \cdot 4}\right) = \frac{1}{3 \cdot 4}\int_0^x t^4\,dt$$

$$= \frac{x^5}{3 \cdot 4 \cdot 5}$$

$\vdots$

$$\boxed{I^n f = \frac{x^{2+n}}{(2+1)(2+2)\cdots(2+n)}}$$