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EX: Compute $\int_0^3 2x^2+x+1\,dx$ means area under $2x^2+x+1$ over $[0,3]$

$f(x)=2x^2+x+1$

$[a,b]=[0,3]$

Soln: $\rightarrow \Delta x = \frac{3-0}{n} = \frac{3}{n}$

$\rightarrow x_k = 0 + k\Delta x = \frac{3k}{n}$

$$\int_0^3 2x^2+x+1\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(2\left(\underbrace{\frac{3k}{n}}_{\frac{9k^2}{n^2}}\right)^2 + \frac{3k}{n} + 1 \right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{18k^2}{n^2}\left(\frac{3}{n}\right) + \frac{3k}{n}\left(\frac{3}{n}\right) + \frac{3}{n} \right]$$

$$= \lim_{n\to\infty} \left[\frac{54}{n^3} \underbrace{\sum_{k=1}^{n} k^2}_{\frac{n(n+1)(2n+1)}{6}} + \frac{9}{n^2} \underbrace{\sum_{k=1}^{n} k}_{\frac{n(n+1)}{2}} + \frac{3}{n} \underbrace{\sum_{k=1}^{n} 1}_{} \right]$$

$1+1+1+\dots+1$
$k=1 \quad k=2$
n of these

2 as $n\to\infty$ 1 as $n\to\infty$

$$= \lim_{n\to\infty} \frac{54}{n^3}\left(\frac{2n^3+\dots}{6}\right) + \frac{9}{n^2}\left(\frac{n^2+n}{\cancel{6}\,2}\right) + 3$$

$$= \frac{54}{6}(2) + \frac{9}{2}(1) + 3 = \frac{51}{2}$$

$$\begin{array}{r} 18 \\ \times 3 \\ \hline 54 \end{array} \qquad \frac{2(27)}{3} + \frac{9}{2} + 3$$

Ex: Look at an example that isn't always positive

$$\int_0^1 -x^2 - 5x - 10\,dx$$

Soln: $[a,b] = [0,1]$

$f(x) = -x^2 - 5x - 10$

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$x_k = 0 + k\Delta x = \frac{k}{n}$

$$f(x_k) = -\left(\frac{k}{n}\right)^2 - 5\frac{k}{n} - 10$$

Compute

$$\int_0^1 -x^2 - 5x - 10\,dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\underbrace{-\left(\frac{k}{n}\right)^2}_{-\frac{k^2}{n^2}} - \frac{5k}{n} - 10\right]\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-\frac{k^2}{n^2}\left(\frac{1}{n}\right) - \frac{5k}{n}\left(\frac{1}{n}\right) - 10\left(\frac{1}{n}\right)\right)$$

$$= \lim_{n\to\infty} \left[-\frac{1}{n^3}\underbrace{\left(\sum_{k=1}^{n} k^2\right)}_{\frac{n(2n+1)(n+1)}{6}} - \frac{5}{n^2}\underbrace{\left(\sum_{k=1}^{n} k\right)}_{\frac{n(n+1)}{2}} - \frac{10}{n}\underbrace{\left(\sum_{k=1}^{n} 1\right)}_{n}\right]$$

$-\frac{1}{3} - \frac{5}{2} - 10$

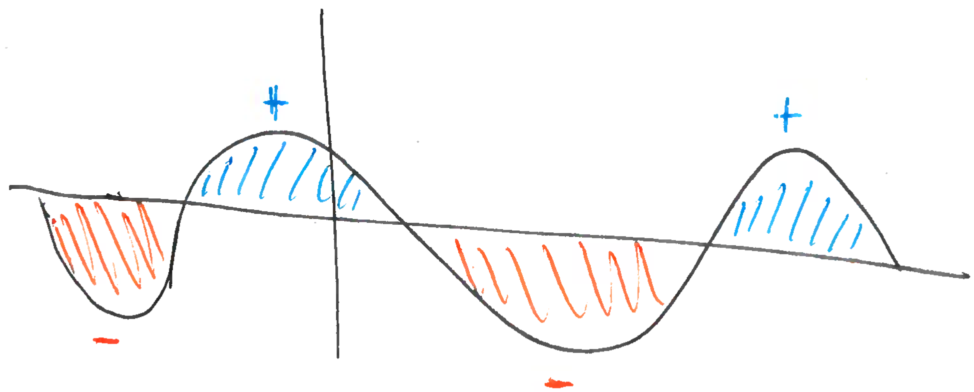
$$= \lim_{n\to\infty} \frac{-2n^3 - \dots}{6n^3} - \frac{5n^2 + 5n}{2n^2} - \frac{10\cancel{n}}{\cancel{n}}$$

$$= -\frac{2}{6} - \frac{5}{2} - 10$$

$$= -\frac{2}{6} - \frac{15}{6} - \frac{60}{6} = -\frac{77}{6}$$

NOT "AREA" b/c Area is never negative...

Integrals actually give us "net area" — meaning areas above x-axis are positive and areas below x-axis are negative.



Add all the $\oplus$ areas and subtract all $\ominus$ areas.
The resulting number is the "net area".

Ex: $\displaystyle\int_{-2}^{2} x\,dx$ ~ $[a,b]=[-2,2]$ $\quad x_k = -2 + k\Delta x = -2 + \frac{4k}{n}$

$$\Delta x = \frac{2-(-2)}{n} = \frac{4}{n} \qquad f(x) = x$$

Compute

$$\int_{-2}^{2} x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-2+\frac{4k}{n}\right)\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{-8}{n} + \frac{16k}{n^2}$$

$$= \lim_{n\to\infty} \left[-\frac{8}{n}\underbrace{\left(\sum_{k=1}^{n} 1\right)}_{=n} + \frac{16}{n^2}\underbrace{\sum_{k=1}^{n} k}_{\frac{n(n+1)}{2}}\right]$$

$$= \lim_{n\to\infty} -\frac{8}{\cancel{n}}(\cancel{n}) + \frac{16}{2n^2}\Big(\overset{n^2+n}{n(n+1)}\Big)$$

$$= -8+8$$

$$= 0$$

Ex: $\int_{-1}^{2} x^3\,dx$ $[a,b]=[-1,2]$ $f(x)=x^3$

$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$

$x_k = -1 + k\Delta x = -1 + \frac{3k}{n}$

$\frac{2^4}{4} - \frac{1}{4}$

Compute

$$\int_{-1}^{2} x^3\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-1+\frac{3k}{n}\right)^3 \left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\frac{27k^3}{n^3} - \frac{27k^2}{n^2} + \frac{9k}{n} - 1\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{27}{n^3}\left(\frac{3}{n}\right)\underbrace{\left(\sum_{k=1}^{n} k^3\right)}_{\frac{n^2(n+1)^2}{4}} - \left(\frac{27}{n^2}\right)\left(\frac{3}{n}\right)\underbrace{\left(\sum_{k=1}^{n} k^2\right)}_{\frac{n(n+1)(2n+1)}{6}} + \frac{9}{n}\left(\frac{3}{n}\right)\underbrace{\left(\sum_{k=1}^{n} k\right)}_{\frac{n(n+1)}{2}} - \frac{3}{n}\underbrace{\left(\sum_{k=1}^{n} 1\right)}_{n}$$

$$= \lim_{n\to\infty} \left(\frac{81}{n^4}\left(\frac{n^4+\dots}{4}\right) - \frac{81}{n^3}\left(\frac{2n^3+\dots}{6}\right) + \frac{27}{n^2}\left(\frac{n^2+n}{2}\right) - 3\right)$$

$$= \frac{81}{4} - \frac{2\cdot 81}{6} + \frac{27}{2} - 3$$

$$= \frac{15}{4}$$