1

<u>Goal</u>: compute $\underbrace{\text{area under curve } 1+x^2 \text{ from } x=0 \text{ to } x=2}_{\int_0^2 1+x^2\,dx}$

<u>facts</u>: $\sum(a+b) = (\sum a) + (\sum b)$

By definitions: $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ $\qquad f(x) = 1+x^2$

$x_k = 0 + k\Delta x = \dfrac{2k}{n}$ $\qquad f(x_k) = 1 + x_k^2$

$2(a+b+c)$

$= 2a+2b+2c$

So, calculate

$$\int_0^2 1+x^2\,dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \frac{4k^2}{n^2}\right)\frac{2}{n}$$

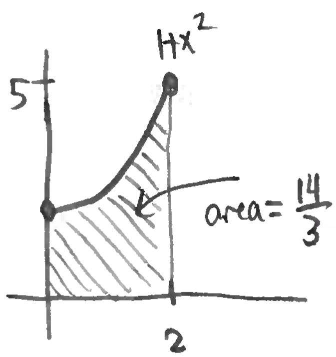
$$= \lim_{n\to\infty}\left[\left(\sum_{k=1}^{n} \frac{2}{n}\right) + \left(\sum_{k=1}^{n} \frac{8k^2}{n^3}\right)\right]$$

no k

no k

variable of sum is "k" ~ all other things are treated like a constant as far as "$\sum$" symbol is concerned

$$\sum_{k=1}^{3} 1 = \underset{(k=1)}{1} + \underset{(k=2)}{1} + 1 = 3$$

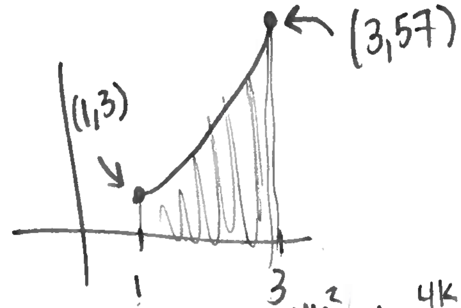


$$= \lim_{n\to\infty}\left[\frac{2}{n}\underbrace{\left(\sum_{k=1}^{n} 1\right)}_{=\underbrace{1+1+1+\dots+1}_{n \text{ times}}} + \frac{8}{n^3}\underbrace{\left(\sum_{k=1}^{n} k^2\right)}_{=\frac{n(n+1)(2n+1)}{6}}\right]$$

$$= \lim_{n\to\infty}\left(\frac{2}{\cancel{n}}\right)\overset{=n}{(\cancel{n})} + \frac{8(\cancel{n})(n+1)(2n+1)}{6n^3} = 2 + \frac{4}{3}\underbrace{\lim_{n\to\infty}\frac{2n^3 + \overset{\text{lower order terms}}{\dots}}{n^3}}_{=2}$$

$$= 2 + \frac{8}{3} = \frac{14}{3}$$

Ex: Find area under curve $2x^3+x$ between $x=1$ and $x=3$.



$\frac{2}{4}x^4+\frac{1}{2}x^2$

$$\left(\tfrac{1}{2}(3^4)+\tfrac{1}{2}(9)\right)-\left(\tfrac{1}{2}+\tfrac{1}{2}\right)=\frac{81}{2}+\frac{9}{2}-\frac{2}{2}=\frac{88}{2}=44$$

$$\left(1+\frac{2k}{n}\right)\left(1+\frac{2k}{n}\right)=1+\frac{2k}{n}+\frac{2k}{n}+\frac{4k^2}{n^2}=1+\frac{4k}{n}+\frac{4k^2}{n^2}$$

$$\left(1+\frac{2k}{n}\right)^3=\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)\left(1+\frac{2k}{n}\right)=1+\frac{4k}{n}+\frac{4k^2}{n^2}+\frac{2k}{n}+\frac{8k^2}{n^2}+\frac{8k^3}{n^3}$$

$$=1+\frac{6k}{n}+\frac{12k^2}{n^2}+\frac{8k^3}{n^3}$$

$a=1,\ b=3$

Soln: $\Delta x=\frac{3-1}{n}=\frac{2}{n}$

$x_k=1+k\Delta x=1+\frac{2k}{n}$

$f(x)=2x^3+x$

$$\int_1^3 f(x)\,dx \overset{\text{def}}{=} \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\,\Delta x$$

$$f(x_k)=2\left(1+\frac{2k}{n}\right)^3+\left(1+\frac{2k}{n}\right)$$
$$=2+\frac{12k}{n}+\frac{24k^2}{n^2}+\frac{16k^3}{n^3}+1+\frac{2k}{n}$$
$$=3+\frac{14k}{n}+\frac{24k^2}{n^2}+\frac{16k^3}{n^3}$$

no k

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(3+\frac{14k}{n}+\frac{24k^2}{n^2}+\frac{16k^3}{n^3}\right)\left(\frac{2}{n}\right)$$

$$=\lim_{n\to\infty}\left[\frac{2}{n}\left(\sum_{k=1}^{n}3\right)+\frac{2}{n}\left(\sum_{k=1}^{n}\frac{14k}{n}\right)+\frac{2}{n}\left(\sum_{k=1}^{n}\frac{24k^2}{n^2}\right)+\frac{2}{n}\left(\sum_{k=1}^{n}\frac{16k^3}{n^3}\right)\right]$$

no k

$$=\lim_{n\to\infty}\left[\frac{6}{n}\left(\sum_{k=1}^{n}1\right)+\frac{28}{n^2}\left(\sum_{k=1}^{n}k\right)+\frac{48}{n^3}\left(\sum_{k=1}^{n}k^2\right)+\frac{32}{n^4}\left(\sum_{k=1}^{n}k^3\right)\right]$$

$$=\lim_{n\to\infty}\frac{6}{\cancel{n}}(\cancel{n})+\frac{28}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{48}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{32}{n^4}\left(\frac{n^2(n+1)^2}{4}\right)$$

$$=6+\lim_{n\to\infty}14\left(\frac{n^2+\dots}{n^2}\right)+8\left(\frac{2n^3+\dots}{n^3}\right)+8\left(\frac{n^4+\dots}{n^4}\right)$$

$\to 1 \qquad \to 2 \qquad \to 1$

$$=6+14+16+8=44$$

Why is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$?

1st if $n=1$:

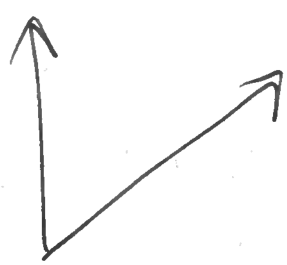
$$\sum_{k=1}^{1} k \overset{?}{=} \frac{1(2)}{2}$$

$$1 = 1 \checkmark$$

if $n=2$:

$$\sum_{k=1}^{2} k \overset{?}{=} \frac{2(3)}{2}$$

$1+2=3$ true!



this style of argument is called "induction"

2nd "induction hypothesis"

Suppose that for some N,

$$\boxed{\sum_{k=1}^{N} k = \frac{N(N+1)}{2}} \quad (\star)$$

Goal: establish truth of

$$\sum_{k=1}^{N+1} k = \frac{(N+1)(N+2)}{2}$$

Calculate:

$$\sum_{k=1}^{N+1} k = (N+1) + \sum_{k=1}^{N} k$$

$$\overset{(\star)}{=} (N+1) + \frac{N(N+1)}{2}$$

$$= \frac{2N+2+N^2+N}{2}$$

$$= \frac{N^2+3N+2}{2}$$

$$= \frac{(N+2)(N+1)}{2}, \text{ as desired!}$$