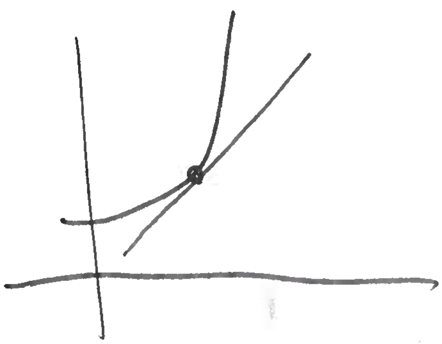
Calculus

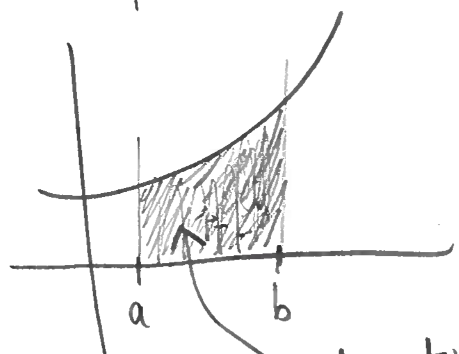
tangent line problem



derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

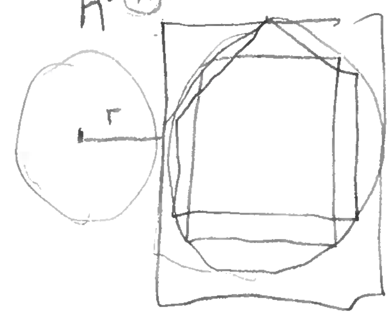
area problem



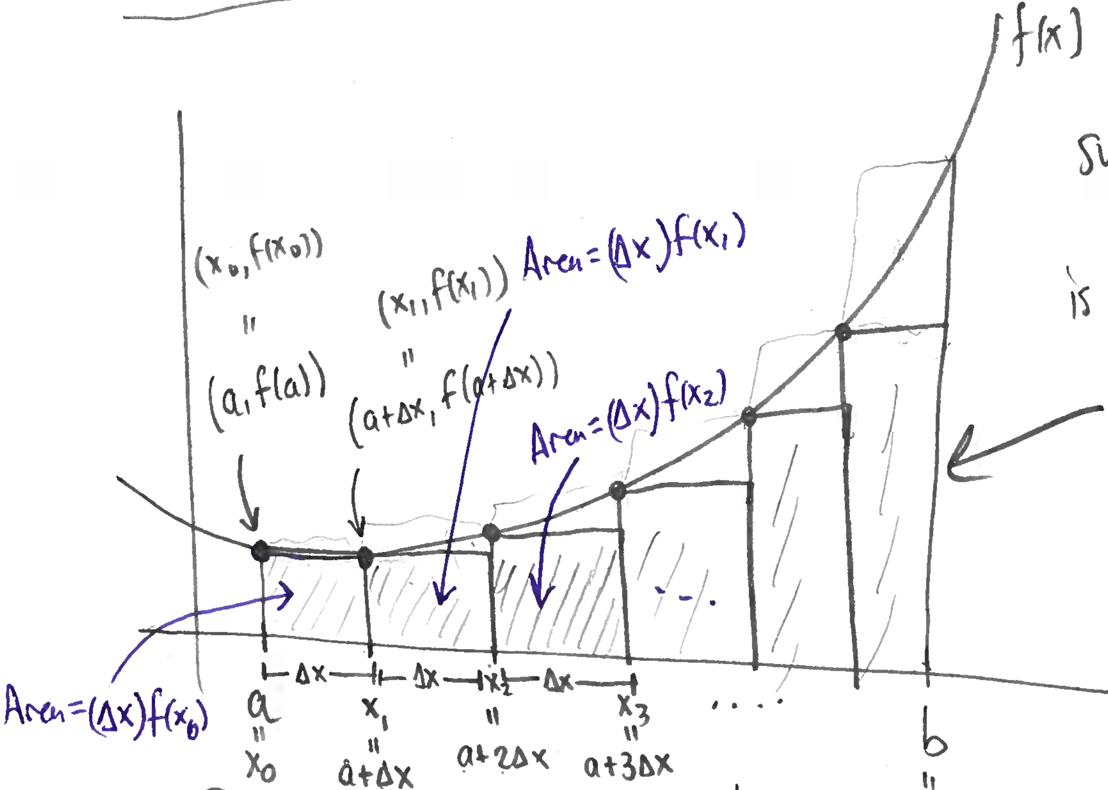
how to find this area?

integrals

$A = \pi r^2$



Our approach to area problem



sum of areas of rectangles is called a Riemann sum

① fix a width ~ $\Delta x = \frac{b-a}{n}$ ← n: number of rectangles you want

② area of each rectangle can be computed + summed:

$$\boxed{RS = \underbrace{f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_{n-1})\Delta x}_{\text{Riemann sum}}}$$

Aside: summation notation:

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + \dots + f(b-1) + f(b)$$

$$\sum_{j=5}^{8} (2j+1) = \underbrace{(2(5)+1)}_{j=5} + \underbrace{(2(6)+1)}_{j=6} + \underbrace{(2(7)+1)}_{j=7} + \underbrace{(2(8)+1)}_{j=8}$$

$$= 11 + 13 + 15 + 17 = 56$$

With summation notation, we can express the Riemann Sum nicely as

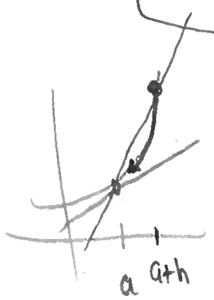
$$\text{Riemann Sum} = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x$$

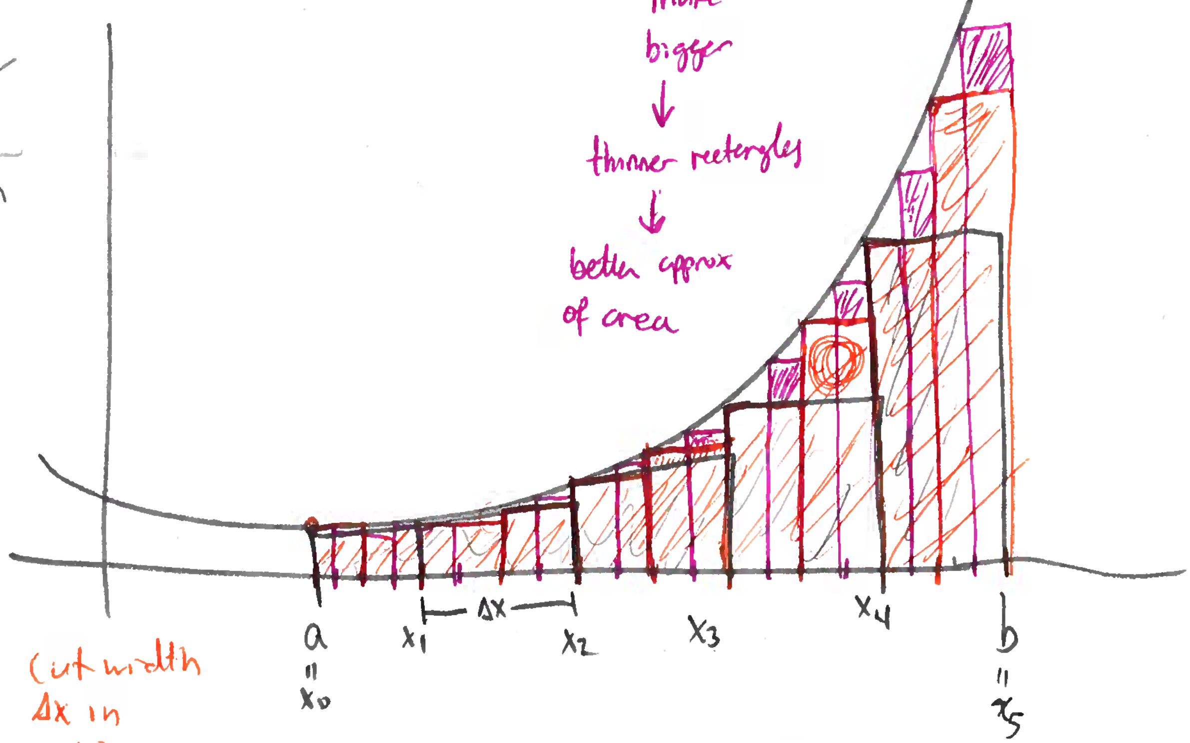
big

$$= \sum_{k=0}^{n-1} f(x_k)\Delta x \quad ; \quad x_k = a + k\Delta x, \quad \Delta x = \frac{b-a}{n}$$

small

Make n bigger → thinner rectangles → better approx of area





Cut width Δx in half

$$\text{Area Under } f(x) \text{ from } a \text{ to } b = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\Delta x, \quad \begin{cases} x_k = a + k\Delta x \\ \Delta x = \frac{b-a}{n} \end{cases}$$

write this w/ notation

$$\int_a^b f(x)\,dx$$

Detour about summations

Gauss ~ add up all numbers b/w 1 and 100

$$1 + 2 + 3 + 4 + \cdots + 96 + 97 + 98 + 99 + 100$$

101

101

101

↑

there are 50 of these

$$\begin{array}{r} 101 \\ \times\ 50 \\ \hline 5050 \end{array}$$

$$\Rightarrow 1 + \cdots + 100 = 50(101) = 5050$$

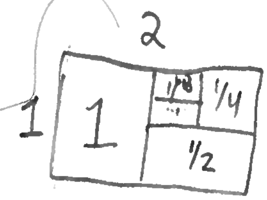
$$\sum_{k=1}^{100} k = 1 + 2 + 3 + \cdots + 100 = 5050$$

In general, $n = 100$: $\frac{100(101)}{2} =$

$$\boxed{\sum_{k=1}^{n} k = \frac{n(n+1)}{2}}$$

$$\sum_{k=1}^{n} k^2 = 1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$



$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$= \frac{\pi^2}{6}$$

Euler (soln of "Basel problem")