Ex: Solve the initial value problem

$$y''(x) = x^2$$

with ICs $y(0)=1,\ y'(0)=3.$

Soln: Recall $y' = D^{-1}y'' = D^{-1}[x^2] = \frac{x^3}{3} + C$

$$y = D^{-1}y' = D^{-1}\left[\frac{x^3}{3} + C\right] = \frac{x^4}{12} + Cx + D$$

Apply my IC's:

$$\underset{\text{given}}{1} = y(0) = \underset{\text{computed}}{\frac{0^4}{12}} + C(0) + D$$

$$\boxed{1 = D}$$

$$\underset{\text{given}}{3} = y'(0) = \underset{\text{computed}}{\frac{0^3}{3}} + C \rightarrow \boxed{C = 3}$$

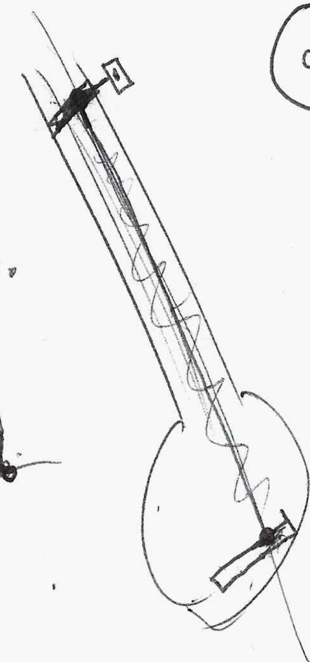
$\Rightarrow$ Soln is

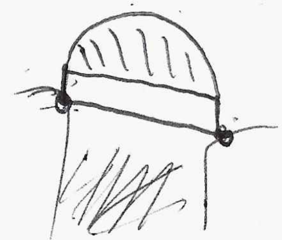
$$\boxed{y(x) = \frac{x^4}{12} + 3x + 1}$$

2

Ex: Solve the boundary value problem

$$y''(t) = e^{2t} \qquad y(0) = 4, \quad y(5) = 6.$$





Soln: $y' = D^{-1}(y'') = D^{-1}(e^{2t}) = \frac{1}{2}e^{2t} + C$

$y(t) = D^{-1}(y') = D^{-1}\left(\frac{1}{2}e^{2t} + C\right) = \frac{1}{2}D^{-1}(e^{2t}) + D^{-1}(C)$

$= \frac{1}{4}e^{2t} + Ct + D$

$4 = y(0) = \frac{1}{4}e^{0} + \cancel{C(0)} + D \rightarrow 4 = \frac{1}{4} + D \rightarrow D = 4 - \frac{1}{4} = \boxed{\frac{15}{4} = D}$

given computed

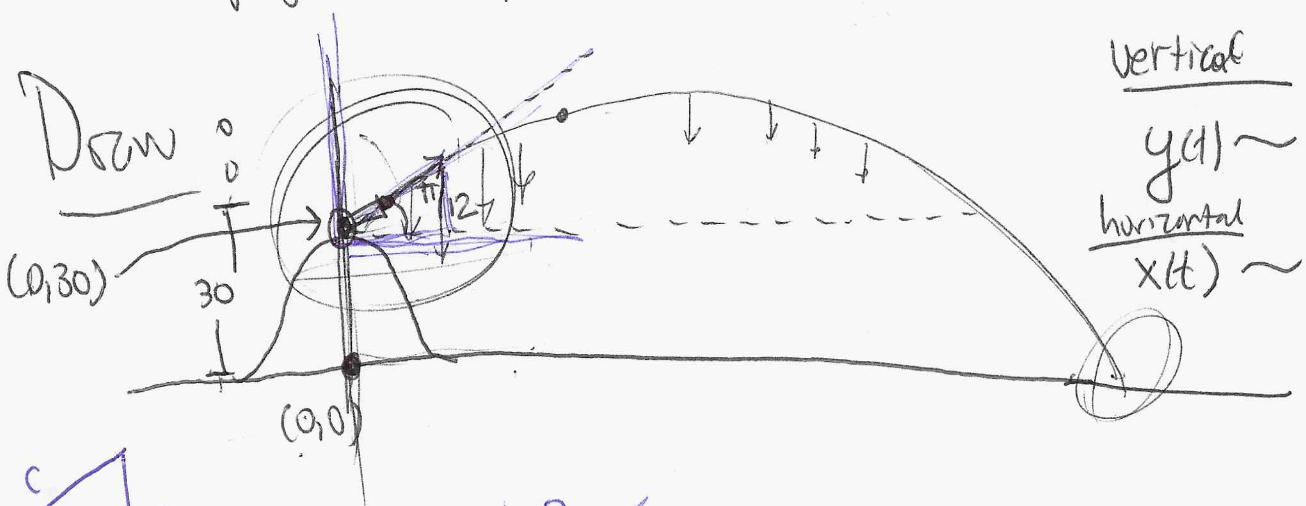
$6 \stackrel{?}{=} y(5) \stackrel{?}{=} \frac{1}{4}e^{10} + 5C + \frac{15}{4} \rightarrow \boxed{C = \frac{6 - \frac{1}{4}e^{10} - \frac{15}{4}}{5}} \approx -1100.87$

$\Rightarrow \boxed{y(t) = \frac{1}{4}e^{2t} - 1100.87\,t + \frac{15}{4}}$

**Ex:** Recall: accel due to gravity (on Earth) $-9.81\,\frac{\text{meters}}{s^2}$ vertical accel.

Spz you fire an artillery cannon from a hill ~~100ft~~ 30m high at an angle ~~of 15°~~ $\frac{\pi}{12}$ rad ~~[illegible]~~ angle of elevation, with initial velocity of projectile of $100\,\frac{m}{s}$. Find eq$^t$ of motion of the projectile. (Assume no wind resistance)

Draw:




Vertical

$y(t) \sim$ given $y''(t) = -9.81$

$y(0) = 30$

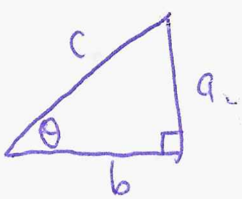
$y'(0) = \underline{\quad}$

horizontal

$x(t) \sim$ given $x''(t) = 0$

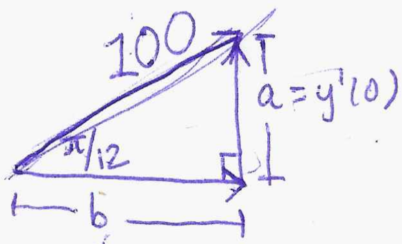
$x(0) = 0$

$x'(0) = \underline{\quad}$




$\sin\theta = \frac{a}{c}$

$\cos\theta = \frac{b}{c}$




$b = x'(0)$

$\Rightarrow$ ~~[illegible]~~

$$\sin\left(\tfrac{\pi}{12}\right) = \frac{y'(0)}{100} \rightarrow y'(0) = 100\sin(\pi/12)$$

$$\cos\left(\tfrac{\pi}{12}\right) = \frac{x'(0)}{100} \rightarrow x'(0) = 100\cos(\pi/12)$$

# We have two IVPs:

$$\begin{cases} y''(t) = -9.81 \\ y(0) = 30, \; y'(0) = 100\sin\left(\frac{\pi}{12}\right) \end{cases}$$

$$\downarrow$$

$$\left[\begin{array}{l} y'(t) = D^{-1}(-9.81) = -9.81t + C \\ y(t) = D^{-1}(y'(t)) = D^{-1}(-9.81t + C) \\ \qquad = \frac{-9.81t^2}{2} + Ct + D \end{array}\right]$$

$$30 \underset{\text{given}}{=} y(0) \underset{\text{computed}}{=} 0 + C(0) + D \Rightarrow \boxed{D = 30}$$

$$100\sin\left(\frac{\pi}{12}\right) \overset{\text{given}}{=} y'(0) \overset{\text{computed}}{=} 0 + C \rightarrow \boxed{C = 100\sin\left(\frac{\pi}{12}\right)}$$

$$\Rightarrow \boxed{y(t) = -\frac{9.81}{2}t^2 + 100\sin\left(\frac{\pi}{12}\right)t + 30}$$

$$\begin{cases} x''(t) = 0 \\ x(0) = 0, \; x'(0) = 100\cos\left(\frac{\pi}{12}\right) \end{cases}$$

$$\downarrow$$

$$\left[\begin{array}{l} x'(t) = D^{-1}(x'') = D^{-1}(0) = C \\ x(t) = D^{-1}(x'(t)) = D^{-1}(C) = Ct + D \end{array}\right]$$

$$0 \underset{\text{given}}{=} x(0) \underset{\text{computed}}{=} D$$

$$100\cos\left(\frac{\pi}{12}\right) \overset{\text{given}}{=} x'(0) \overset{\text{computed}}{=} C$$

$$\Rightarrow \boxed{x(t) = 100\cos\left(\frac{\pi}{12}\right)t}$$

Q1: How long until projectile hits ground?

Q2: How much horizontal distance does the projectile travel before it hits ground?

Q1: Solve $y(t) = 0$

$$-\frac{9.81}{2}t^2 + 100\sin\left(\frac{\pi}{12}\right)t + 30 = 0$$

$$-4.91t^2 + 25.88t + 30 = 0$$

$$t = \frac{-25.88 \pm \sqrt{(25.88)^2 - 4(-4.91)(30)}}{(-4.91)(2)}$$

$+$ $t = -0.978$ ← time before we launch projectile — physically meaningless

$-$ $t = 6.25$ ← time it lands

seconds after launch

Q2: Plug $t = 6.25$ into $x$:

$$x(6.25) = 100\cos\left(\frac{\pi}{12}\right)(6.25)$$

$$\approx 603.7\,m$$