Ex: $\lim_{x\to\infty} \frac{x\ln(x)}{x^2} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[x\ln(x)]}{2x}$ $\frac{x}{x}$ $\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{\ln(x)+1}{2x}$ $\frac{\infty}{\infty}$

$\overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2}$

$= \frac{1}{2}\lim_{x\to\infty} \frac{1}{x} = 0$

$x\ln(x) << x^2$

$\ln(x) << x << x\ln(x) << x^2 << x^3 << \dots << e^x << x^x$

Ex: Stirling approximation – tells us growth rate of (factorial function)

"empty sum" ⇝ additive identity ⇝ 0

"empty prod" ⇝ mult. identity → 1

$\frac{24}{\times 5}{120}$ $\boxed{0! = 1}$

$3! = 3\cdot2\cdot1 = 6$

$4! = 4\cdot3\cdot2\cdot1 = 24$

$5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$

$n! = n(n-1)\dots(3)(2)(1)$

$\text{fac}(0) := 1$
$\text{fac}(n) = \text{fac}(n-1)\cdot n$

$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

"approx = to"

formally

$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$

Compute

$$\lim_{n\to\infty} \frac{e^n}{n!} \overset{\text{Stirling}}{=} \lim_{n\to\infty} \frac{e^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

$$= \lim_{n\to\infty} \frac{e^{2n}}{\sqrt{2\pi n}\; n^n} = 0$$

$e^{2n} = (e^2)^n$

$(a^b)^c = e^{bc}$

$e^2 \approx 8.\_\_$

($n!$ faster then e^n)

$$\lim_{n\to\infty} \frac{n^n}{n!} \overset{\text{Stirling}}{=} \lim_{n\to\infty} \frac{n^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^n} e^n}{\sqrt{2\pi n}\; \cancel{n^n}}$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{e^n}{\sqrt{n}} = \infty$$

$1 \ll \sqrt{x} \ll x$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$5^5 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

Conclusion: $\boxed{n^n \text{ faster then } n!}$

Ex: $$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = 2\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

$\frac{\infty}{\infty}$

$\sqrt{x} = x^{1/2}$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$

$$1 \ll \ln(x) \ll x^p \ll x \ll x\ln(x) \ll x^2 \ll x^3 \ll \ldots \ll e^x \ll x! \ll x^x$$

$0 < p < 1$

Ex: (Prime number theorem) — (1898)

primes ~ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

numbers divisible only by itself + 1

(pos. integers)

more rare as you go bigger

cryptography "RSA"

PNT: growth rate of # of primes

Define:

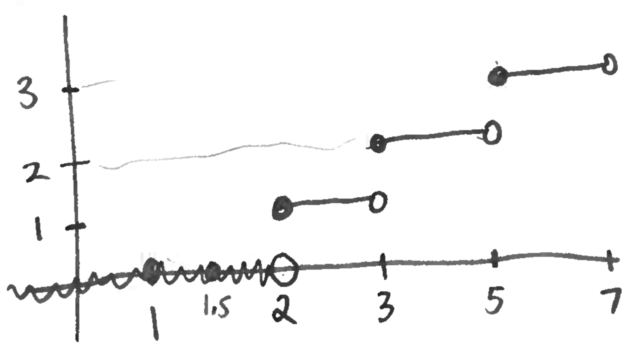
$\pi(x) = \#\text{ primes} \leq x$

$\pi(1)$

$\pi(1.5)$

$\pi(2) = \#\text{ primes} \leq 2$

$\pi(4)$



What is growth rate of $\pi(x)$?

Was shown:

$\boxed{\pi(x) \text{ has growth rate that matches } \frac{x}{\ln(x)}}$

"About how many primes are $\leq$ 5,000,000?"

About $\frac{5{,}000{,}000}{\ln(5{,}000{,}000)} \approx 324{,}000$

Ex: $\lim_{x\to-\infty} \sin\left(\frac{10}{x}\right) = \sin(0) = 0$

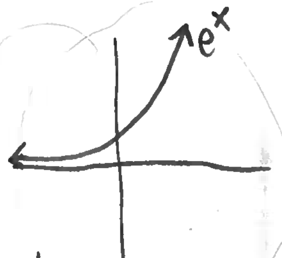
We know $\frac{10}{x}\to 0$ as $x\to-\infty$

$\frac{1}{x}\to 0$
$x\to\pm\infty$

Ex: $\lim_{x\to-\infty}\frac{\overset{\nearrow 0}{e^{2x}}+\overset{\nearrow \infty}{e^{-2x}}}{\underset{\swarrow 0}{e^{2x}}-\underset{\searrow \infty}{e^{-2x}}} \overset{\text{try LH}}{=} \lim_{x\to-\infty}\frac{2e^{2x}-2e^{-2x}}{2e^{2x}+2e^{-2x}}$

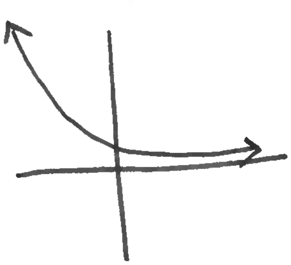
$\frac{\infty}{\infty}$

trouble $\frac{\infty}{\infty}$
doesn't seem that L'H simplified it



$\lim_{x\to\infty} e^{x}=\infty$

$\lim_{x\to-\infty} e^{x}=0$



$\lim_{x\to\infty} e^{-x}=0$

$\lim_{x\to-\infty} e^{-x}=\infty$

Algebra saves the day:

$$\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}} = \frac{e^{2x}+\frac{1}{e^{2x}}}{e^{2x}+\frac{1}{e^{2x}}}$$

$$= \frac{\left(\frac{(e^{2x})^2+1}{e^{2x}}\right)}{\left(\frac{(e^{2x})^2-1}{e^{2x}}\right)} = \left(\frac{e^{4x}+1}{\cancel{e^{2x}}}\right)\left(\frac{\cancel{e^{2x}}}{e^{4x}-1}\right)$$

$$= \frac{e^{4x}+1}{e^{4x}-1}$$

So,

$$\lim_{x\to-\infty}\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}} = \lim_{x\to-\infty}\frac{\overset{0}{e^{4x}}+1}{\underset{0}{e^{4x}}-1} = \frac{0+1}{0-1} = -1$$