Ex: $$\lim_{x\to\infty} \frac{3x^3-2x^2+x+1}{x^3+x^2-5}$$

$$= \lim_{x\to\infty} \frac{3x^3-2x^2+x+1}{x^3+x^2-5}\left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right)$$

$$= \lim_{x\to\infty} \frac{3-\frac{2}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{1}{x}-\frac{5}{x^3}}$$

$$= \frac{3}{1} = 3$$

FACT for any $a>0$,

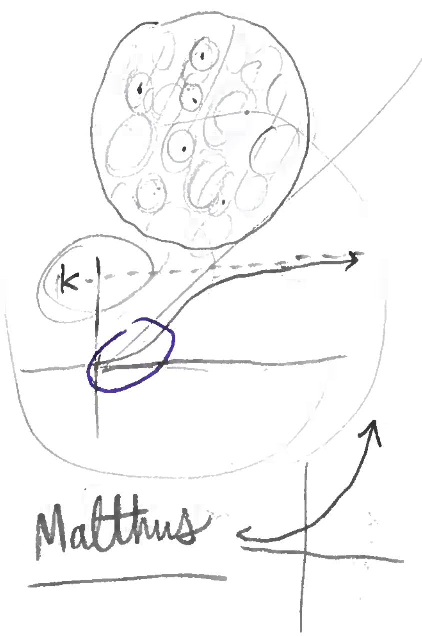
$$\lim_{x\to\infty} \frac{1}{x^a} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0, \quad n=1,2,3,\dots$$

Ex: $$\lim_{x\to-\infty} \frac{x^2-3x+2}{x^3+1} = \lim_{x\to-\infty} \left(\frac{x^2-3x+2}{x^3+1}\right)\left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to-\infty} \frac{\frac{1}{x}-\frac{3}{x^2}+\frac{2}{x^3}}{1+\frac{1}{x^3}}$$

$$= \frac{0}{1} = 0$$



Malthus

Ex: $$\lim_{x\to\infty} \frac{x^3-x+1}{6000x^2+x-2}$$

$$= \lim_{x\to\infty} \frac{x^3-x+1}{6000x^2+x-2}\underbrace{\left(\frac{\frac{1}{x^3}}{1/x^3}\right)}_{=1}$$

$$= \lim_{x\to\infty} \frac{1-\frac{1}{x^2}+\frac{1}{x^3}}{\frac{6000}{x}+\frac{1}{x^2}-\frac{2}{x^3}} = \frac{1}{0} = +\infty$$

Hierarchy of function growth rates

$$1 \ll x \ll x^2 \ll x^3 \ll \ldots \ll x^n \ll x^{n+1} \ll \ldots$$

grows much faster than this one

Algorithmic complexity ~ $O(n \log n)$

Bernoulli

L'Hôpital's Rule

L'Hospital's Rule

allows us to compare growth rates using derivatives

$$\frac{x^3+x}{x^2+1}$$

$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, 1^\infty, \infty-\infty, 0^0, \infty^0,$ etc

Theorem (L'Hôpital's Rule):

If $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is of indeterminate form,

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

Ex: $\lim\limits_{x\to\infty} \frac{3x^3-2x^2+x+1}{x^3+x^2-5} \overset{\text{L.H.}}{=} \lim\limits_{x\to\infty} \frac{9x^2-4x+1}{3x^2+2x}$ $\frac{\infty}{\infty}$

$\frac{\infty}{\infty}$

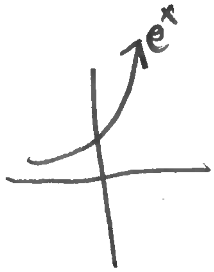
$\overset{\text{L.H.}}{=} \lim\limits_{x\to\infty} \frac{18x-4}{6x+2}$ $\frac{\infty}{\infty}$

$\overset{\text{L.H.}}{=} \lim\limits_{x\to\infty} \frac{18}{6} = \frac{18}{6} = 3$

Ex: $\lim\limits_{x\to\infty} \dfrac{e^x}{x^3+1} \overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{3x^2} \overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{6x}$

$\frac{\infty}{\infty}$ $\quad$ $\frac{\infty}{\infty}$

$\overset{LH}{=} \lim\limits_{x\to\infty} \dfrac{e^x}{6} = \infty$



$\Rightarrow$ e^x grows faster than x^3

FACT: can show e^x grows faster than x^n for any n

$\cancel{x} \ll x^2 \ll \dots \ll x^n \ll x^{n+1} \ll \dots \ll \dots \ll e^x$

Ex: $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{2x} = \lim\limits_{x\to\infty} \dfrac{1}{2x^2} = 0$

$\frac{\infty}{\infty}$

FACT: can show $\ln(x)$ grows slower than any polynom

$\ln(x) \ll x \ll x^2 \ll \dots \ll e^x \ll x^x$

Who wins? $x\ln(x)$ or x^2?

Ex: $\lim\limits_{x\to\infty} \dfrac{e^x}{x^x} = \lim\limits_{x\to\infty} \dfrac{e^x}{[1+\ln(x)]x^x} \overset{\text{should}}{=} 0$

$\frac{\infty}{\infty}$

$\dfrac{d}{dx} x^x = \dfrac{d}{dx} e^{x\ln(x)}$

$= x^x\left[\ln(x) + \frac{x}{x}\right]$

$= [1+\ln(x)]x^x$