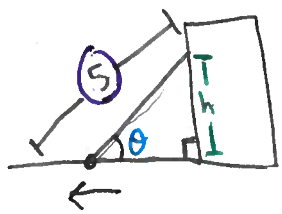
Ex: A ladder leans against a vertical wall. The ladder is 5ft long & top of it is falling down wall at a rate of $\frac{1}{2}\,\frac{ft}{s}$. How is angle between the bottom of ladder & ground changing when the top of ladder is 2ft from ground?

$\frac{dh}{dt} = -\frac{1}{2}$

$\frac{d\theta}{dt}$

$h=2$

Soln:



Given: $\frac{dh}{dt} = -\frac{1}{2}$

Goal: $\frac{d\theta}{dt}$ when $h=2$

$$\sin(\theta) = \frac{h}{5}$$

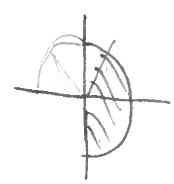
$$\Big\downarrow \frac{d}{dt}$$

$$\frac{d}{dt}\sin(\theta) = \frac{d}{dt}\left(\frac{h}{5}\right)$$

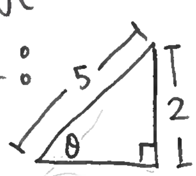
$$\frac{d\theta}{dt}\frac{d}{d\theta}\sin(\theta) = \frac{1}{5}\frac{dh}{dt}$$

$$\frac{d\theta}{dt}\cos(\theta) = \frac{1}{5}\left(-\frac{1}{2}\right) \longrightarrow \boxed{\frac{d\theta}{dt} = -\frac{1}{10\cos(\theta)}}$$

$\arcsin(\sin\theta) = \theta$



If $h=2$:



$\sin(\theta) = \frac{2}{5}$

$\boxed{\theta = \arcsin\left(\frac{2}{5}\right)}$

So, $$\left.\frac{d\theta}{dt}\right|_{h=2} = \frac{-1}{10\underbrace{\cos\left(\arcsin\left(\frac{2}{5}\right)\right)}_{\text{exact value}}}$$

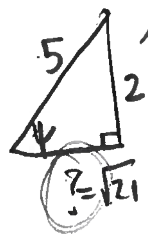
②

We got
$$\left.\frac{d\theta}{dt}\right|_{h=2} = \frac{-1}{10\cos\left(\arcsin\left(\frac{2}{5}\right)\right)} = \frac{-1}{10\left(\frac{\sqrt{21}}{5}\right)} = -\frac{1}{2\sqrt{21}}\ \frac{\text{rad}}{\text{sec}}$$

$$\cos\Big(\underbrace{\arcsin\left(\tfrac{2}{5}\right)}_{\psi}\Big) = \cos(\psi) = \frac{\sqrt{21}}{5}$$

$\psi = \arcsin\left(\frac{2}{5}\right)$

$\sin(\psi) = \frac{2}{5}$



$?^2 + 2^2 = 5^2$

$? = \sqrt{21}$

Ex: Same problem done differently

$$\sin(\theta) = \frac{h}{5}$$
$$\downarrow$$
$$\theta = \arcsin\left(\frac{h}{5}\right)$$
$$\downarrow \frac{d}{dt}$$

Recall
$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d\theta}{dt} = \frac{d}{dt}\arcsin\left(\frac{h}{5}\right)$$

$$\frac{d\theta}{dt} = \frac{d\left(\frac{h}{5}\right)}{dt}\ \frac{d}{d\left(\frac{h}{5}\right)}\arcsin\left(\frac{h}{5}\right)$$

$$= \frac{1}{5}\frac{dh}{dt}\ \frac{1}{\sqrt{1-\left(\frac{h}{5}\right)^2}} = -\frac{1}{10}\ \frac{1}{\sqrt{1-\left(\frac{h}{5}\right)^2}}$$

$$\sqrt{1-\frac{4}{25}} = \sqrt{\frac{25}{25}-\frac{4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$$

So, $$\left.\frac{d\theta}{dt}\right|_{h=2} = -\frac{1}{10}\ \frac{1}{\sqrt{1-\left(\frac{2}{5}\right)^2}} = -\frac{1}{10}\ \frac{1}{\sqrt{1-\frac{4}{25}}} = -\frac{1}{10}\ \frac{1}{\sqrt{21/25}} = -\frac{1}{10}\ \frac{1}{\frac{\sqrt{21}}{5}} = -\frac{1}{10}\cdot\frac{5}{\sqrt{21}}$$
$$= -\frac{1}{2\sqrt{21}}\ \frac{\text{rad}}{\text{s}}$$

Why use radians in calculus?

$$360^\circ = 1 \text{ full rotation} = 2\pi \text{ rad}$$

div by 360°

$$\boxed{1 = \frac{2\pi \text{ rad}}{360^\circ}}$$

div by 2π rad

$$\boxed{\frac{360^\circ}{2\pi \text{ rad}} = 1}$$

$$x^\circ = (x^\circ)(1) = x^{\cancel{\circ}}\left(\frac{2\pi \text{ rad}}{360\cancel{^\circ}}\right) = \frac{2\pi x}{360}\text{ rad} = \frac{x\pi}{180}\text{ rad}$$

$$\frac{d}{dx}\sin(x^\circ) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right)$$

mismatch

$$= \frac{d\left(\frac{\pi x}{180}\right)}{dx}\,\frac{d}{d\left(\frac{\pi x}{180}\right)}\sin\left(\frac{\pi x}{180}\right)$$

$$= \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right)$$

$$= \frac{\pi}{180}\cos(x^\circ)$$

AWFUL

extra factor when using degrees

Ex: When air expands adiabatically, its pressure P and volume V are related by

$$PV^{1.4} = C,$$

where C is constant. Spz at some time, the volume is 300 cm^3 and pressure is 76 kPa + is decreasing at rate of 8 $\frac{\text{kPa}}{\text{min}}$. At what rate is volume changing at that time?

Soln: Given: $\frac{dP}{dt} = -8$

Goal: $\frac{dV}{dt}$ when $P=76$, $V=300$

$(fg)' = f'g + g'f$

$$PV^{1.4} = C$$

$$\Big\downarrow \frac{d}{dt}$$

$$\underbrace{\frac{d}{dt}\left(PV^{1.4}\right)}_{\text{prod Rule}} = \cancel{\frac{d}{dt} C} \rightarrow 0$$

derivative of constant is zero

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{dP}{dt} V^{1.4} + P \frac{d}{dt} V^{1.4} = 0$$

mismatch

$1.4 - 1 = 0.4$

$$\frac{dP}{dt} V^{1.4} + P \frac{dV}{dt} \frac{d}{dV} V^{1.4} = 0$$

$$-8V^{1.4} + P \boxed{\frac{dV}{dt}} 1.4V^{0.4} = 0$$

$$\frac{dV}{dt} = \frac{8V^{1.4}}{1.4PV^{0.4}} = \frac{8V}{1.4P}$$

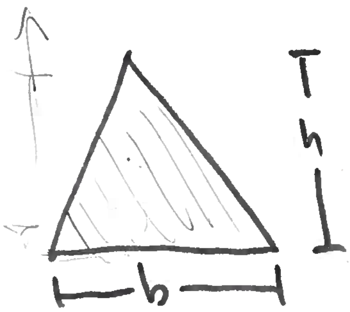
$$\left.\frac{dV}{dt}\right|_{\substack{P=76 \\ V=300}} = \frac{8(300)}{1.4(76)} \approx 22.55 \frac{\text{cm}^3}{\text{min}}$$

Ex: ~~Altitu~~ Height (h) of a Δ is inc. at rate of 4 $\frac{cm}{min}$ $\quad \frac{dh}{dt} = 4$

while area (A) is inc at rate of 1 $\frac{cm^2}{min}$ $\quad \frac{dA}{dt} = 1$ $\qquad$ 5

At what rate is base changing ($\frac{db}{dt}$) when height is 15 cm and ($h = 15$) area is 79 cm²? ($A = 79$)

Soln:



$$\text{Area} = A = \tfrac{1}{2}bh \quad \xrightarrow{d/dt}$$

$$\frac{dA}{dt} = \frac{1}{2}\frac{d}{dt}[bh] \qquad " \frac{d(bh)}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2}\left[\frac{db}{dt}h + b\frac{dh}{dt}\right]$$

$$\frac{dA}{dt} = \frac{1}{2}\frac{db}{dt}h + \frac{b}{2}\frac{dh}{dt}$$

$$\frac{db}{dt} = \frac{1 - \frac{b}{2}\cdot 4}{\frac{1}{2}h} = \frac{2-4b}{h}$$

So, $$\left.\frac{db}{dt}\right|_{h=15,\,A=79} = \frac{2-4b}{15} = \frac{2-4\left(\frac{158}{15}\right)}{15} \approx -2.68\ \frac{cm}{min}$$

$$79 = \tfrac{1}{2}b(15)$$

$$\frac{158}{15} = b$$

$$79 \cdot 2 = 158$$