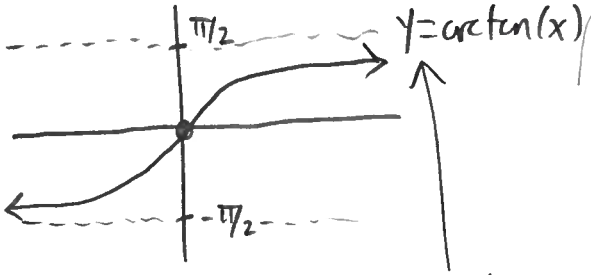
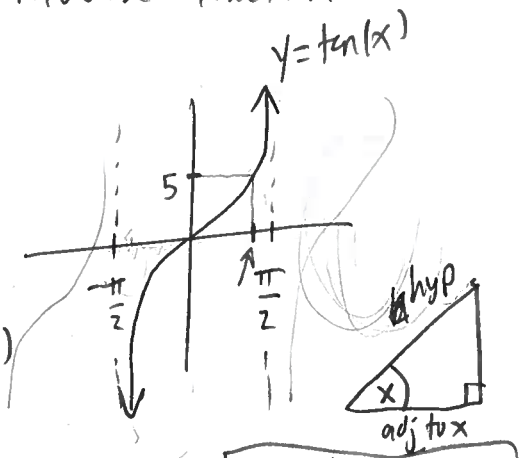
EX: Recall arctan' function — inverse function for tan(x).

arctan: $\mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$




$\sec x = \frac{\text{hyp}}{\text{adj to } x}$

Recall: $\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$ ($\cos^2(x)+\sin^2(x) = 1$)

Soln: if $y = \arctan(x)$

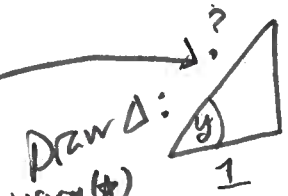
⇓ plug both sides into tan(x)

$\tan(y) = \tan(\arctan(x)) = x$ (inverses)

(★) $\tan(y) = x$

Goal: find $\frac{dy}{dx}$

$\tan y = \frac{\text{opp } y}{\text{adj } y}$




Draw Δ: using (★)

Pythagorean thm → $1^2 + x^2 = ?^2$

$? = \sqrt{1+x^2}$

Take $\frac{d}{dx}$ of (★):

$\frac{d}{dx}\tan(y) = \frac{d}{dx}[x]$

mismatch

$= 1$

$\frac{dy}{dx}\frac{d}{dy}\tan(y) = 1 \longrightarrow \frac{dy}{dx}\sec^2(y) = 1$

$\frac{dy}{dx} = \cos^2(y)$

$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1+x^2}}\right)^2$

$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Ex: Find $\frac{d}{dx}\arccos(x)$. $\cos^{-1}$

Soln: $y = \arccos(x)$

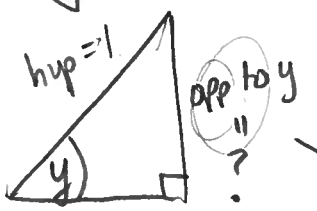
$\cos y = x$

$\downarrow \frac{d}{dx}$

$\frac{d}{dx}\cos(y) = \frac{d}{dx}x$

$-\sin(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = -\frac{1}{\sin(y)}$




Pyth thm

$x^2 + ?^2 = 1^2$

$? = \sqrt{1-x^2}$

$\sin(y) = \frac{\text{app to } y}{\text{hyp}} = \sqrt{1-x^2}$

$$\frac{d}{dx}\arccos(x) = \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Expect: Written HW doing $\frac{d}{dx}\arcsin(x)$

Ex: Find $\frac{d}{dx}\arctan(e^x)$. ← $\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Soln:

$$\frac{d}{dx}\arctan(e^x) = \frac{d(e^x)}{dx}\frac{d}{d(e^x)}\arctan(e^x)$$

mismatch

$$= e^x \frac{1}{1+(e^x)^2}$$

$$= \frac{e^x}{1+e^{2x}}$$

Ex: $\frac{d}{dx}\arccos(\csc(x))$ ← $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$

mm

$$= \frac{d(\csc(x))}{dx}\frac{d}{d(\csc(x))}\arccos(\csc(x))$$

$\frac{d}{dx}\csc(x) = \frac{d}{dx}\frac{1}{\sin(x)}$

$= -\frac{\cos(x)}{\sin^2(x)}$

$= -\cot(x)\csc(x)$

$$= -\cot(x)\csc(x)\left(-\frac{1}{\sqrt{1-\csc^2(x)}}\right)$$

$$= \frac{\cot(x)\csc(x)}{\sqrt{1-\csc^2(x)}}$$

# Rates of change + related rates

"Rate of change" almost always refers to some kind of derivative

**Ex:** freq of vibrations in a violin string is $f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$

$f$ ← frequency; $L$ ← length of string; $T$ ← tension; $\rho$ ← linear density

Find rate of change (derivative) of freq w.r.t. tension ($\frac{df}{dT}$) (when $L$ and $\rho$ are constant)

$\frac{d}{dT}L = 0$ $\frac{d}{dT}\rho = 0$

as tension goes up, freq goes up →

$$\frac{d}{dT}f = \frac{d}{dT}\left[\frac{1}{2L}\sqrt{\frac{T}{\rho}}\right]$$

$$= \frac{1}{2L\sqrt{\rho}}\frac{d}{dT}\left(T^{1/2}\right)$$

$$= \frac{1}{2L\sqrt{\rho}}\left(\frac{1}{2}\right)T^{-1/2} = \frac{1}{4L\sqrt{\rho}\sqrt{T}} > 0$$

freq ~ Hz

tension ~ N

$\frac{df}{dT} \sim \frac{Hz}{N}$

$\frac{df}{dL} \sim \frac{Hz}{inch}$

dist / time ↓ velocity ~ $\frac{m}{s}$

accel ~ $\frac{m}{s^2}$

**Ex:** If $s(t) = -5t^2 + 2t - 1$ Represents position (meters from start pt) + t (# seconds), find velocity + acceleration at time t.

**Soln:** $s(t)$ meters

$s'(t)$ $\frac{\text{meters}}{\text{sec}}$

$s''(t)$ $\frac{\frac{\text{meters}}{\text{sec}}}{\text{sec}} = \frac{\text{meters}}{\text{sec}^2}$

velocity $= v(t) = s'(t) = -10t + 2$

accel $= a(t) = s''(t) = v'(t) = -10$

---

**Ex:** Suppose price ($) related to quantity demanded (units) are related by

$$p^3 + q^2 = 5$$

What is rate of change (derivative) of price (function to diff'ate) w.r.t. quantity (variable on bottom of $\frac{dp}{dq}$)?

✓

$$\frac{d}{dq}p^3 + \frac{d}{dq}q^2 = \frac{d}{dq}5$$

(mismatch)

$$3p^2\frac{dp}{dq} + 2q = 0$$

$$\frac{dp}{dq} = \frac{-2q}{3p^2}$$

Ex: Boyle's Law:

volume

pressure → $PV = C$ ← const

Find r.o.c. of pressure w.r.t. volume

$$\frac{dP}{dV}$$

Soln:

$$\frac{d}{dV}(PV) = \frac{d}{dV}C \quad (\frac{d}{dV}C \to 0)$$

$$\frac{dP}{dV}V + P\frac{dV}{dV} = 0 \quad (\frac{dV}{dV} \to 1)$$

$$\frac{dP}{dV}V + P = 0$$

$$\frac{dP}{dV} = \frac{-P}{V}$$

as vol ↑

pressure ↓