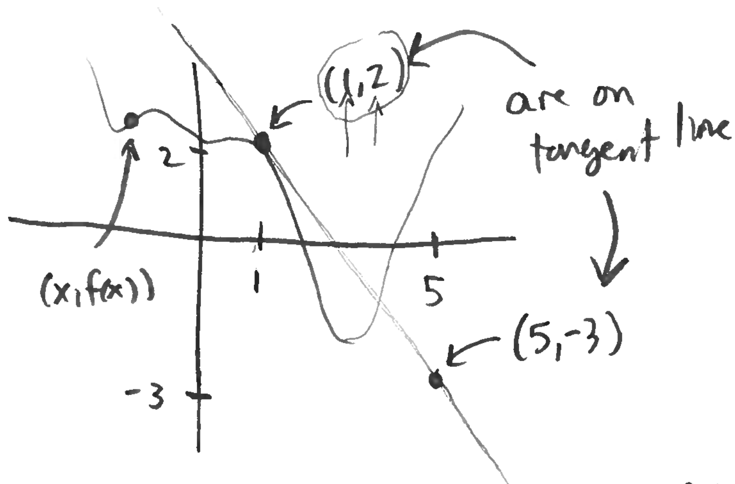
Resembles HW4 Problem 9.

Suppose the tangent line to f at (1,2) passes thru (5,-3). Compute $f(1)$ and $f'(1)$.



slope of tan line at (1,2) $\equiv f'(1)$
(x=1)

$$\begin{cases} f'(1) = \text{slope of line cont (1,2) and } (5,-3) = \dfrac{-3-2}{5-1} = \dfrac{-5}{4} \\ f(1) = 2 \end{cases}$$

all problems below are "implicit differentiation"

Ex: A Kampyle of Eudoxus is given by

(★) $$x^4 = x^2 + y^2$$

Find eqt for tan line at $x=2$ for the point lying in QI.

Soln: Recall: slope is $\frac{dy}{dx}$

Take $\frac{d}{dx}$ of both sides of (★):

$$\frac{d}{dx} x^4 = \frac{d}{dx} x^2 + \boxed{\frac{d}{dx} y^2 \quad \text{mismatch} \rightarrow \text{chain rule}}$$

QII	QI
QIII	QIV

$$4x^3 = 2x + \frac{dy}{dx} \frac{d}{dy} y^2$$

$$4x^3 = 2x + \frac{dy}{dx}(2y)$$

$$\boxed{\frac{dy}{dx} = \frac{4x^3 - 2x}{2y} = \frac{2x^3}{y} - \frac{x}{y}}$$ — slope of tan line at all pts on curve

At $x=2$, (★)

$$2^4 = 2^2 + y^2$$

$$16 - 4 = y^2$$

$$12 = y^2 \rightarrow y = \pm\sqrt{12} \xrightarrow{QI} y = +\sqrt{12}$$

$\Rightarrow (2, \sqrt{12})$ is on curve

Eqt of tan line is

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)=(2,\sqrt{12})} = \frac{2(2^3)}{\sqrt{12}} - \frac{2}{\sqrt{12}} \Longrightarrow y - \sqrt{12} = \left(\frac{14}{\sqrt{12}}\right)(x-2)$$

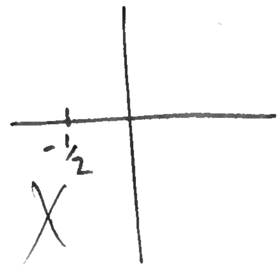
Ex: Bullet-nosed curve

(*) $$y^2 - x^2 = x^2 y^2$$

Find eqt of tan line at $x = -\frac{1}{2}$ in QIII.

Soln: At $x = -\frac{1}{2}$, (*) becomes

$$y^2 - \left(-\tfrac{1}{2}\right)^2 = \left(-\tfrac{1}{2}\right)^2 y^2$$



$$y^2 - \frac{1}{4} = \frac{y^2}{4}$$

$$\frac{3y^2}{4} = \frac{1}{4}$$

$$y^2 = \frac{1}{3} \to y = \pm\frac{1}{\sqrt{3}} \xrightarrow{QIII} y = -\frac{1}{\sqrt{3}}$$

$\Rightarrow \boxed{\left(-\frac{1}{2}, -\frac{1}{\sqrt{3}}\right)}$ on curve

Take $\frac{d}{dx}$ of (*):

$(fg)' = f'g + fg'$

$$\frac{d}{dx} y^2 - \frac{d}{dx} x^2 = \underbrace{\frac{d}{dx} x^2 y^2}_{\text{prod rule}}$$

mismatch

$$\frac{dy}{dx}\frac{d}{dy} y^2 - 2x = y^2\left[\frac{d}{dx} x^2\right] + x^2\left[\frac{d}{dx} y^2\right]$$

same

$$\frac{dy}{dx}\cdot 2y - 2x = 2xy^2 + 2x^2 y \frac{dy}{dx}$$

combine

$2x+1=3x-2$

$$\frac{dy}{dx}\left[2y - 2x^2 y\right] = 2xy^2 + 2x$$

$$\frac{dy}{dx} = \frac{2xy^2 + 2x}{2y - 2x^2 y}$$

Cont...

Slope is

$$\left.\frac{dy}{dx}\right|_{(x,y)=\left(-\frac{1}{2},-\frac{1}{\sqrt{3}}\right)} = \frac{2\left(-\frac{1}{2}\right)\left(-\frac{1}{\sqrt{3}}\right)^2+2\left(-\frac{1}{2}\right)}{2\left(-\frac{1}{\sqrt{3}}\right)-2\left(-\frac{1}{2}\right)^2\left(-\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{-\frac{1}{3}-1}{-\frac{2}{\sqrt{3}}+\frac{2}{4\sqrt{3}}}$$

$\Rightarrow$ tan line is

$$y-\left(-\frac{1}{\sqrt{3}}\right)=\left(\frac{-4/3}{-2/\sqrt{3}+2/4\sqrt{3}}\right)\left(x-\left(-\frac{1}{2}\right)\right)$$

Ex: Given curve $\cos(xy)=\sin(xy)$ find eqt of tan line at $x=y=\frac{\sqrt{\pi}}{2}$

$xy=\left(\frac{\sqrt{\pi}}{2}\right)\left(\frac{\sqrt{\pi}}{2}\right)=\frac{\pi}{4}$

Soln: Verify $x=y=\frac{\sqrt{\pi}}{2}$ is on graph:

$$\cos\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right) \quad \text{TRUE}$$

$$\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}, \quad \sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$$

$\frac{d}{dx}\sin x=\cos x$

Take $\frac{d}{dx}$ to get

$$\frac{d}{dx}\cos(xy)=\frac{d}{dx}\sin(xy)$$

mismatch

By chain rule

$$(\star\star)\quad \frac{d(xy)}{dx}\frac{d}{d(xy)}\cos(xy) = \frac{d(xy)}{dx}\frac{d}{d(xy)}\sin(xy)$$

Match!

prod rule

$$\frac{d}{dx}(xy) = \underbrace{\left(\frac{d}{dx}[x]\right)}_{=1} y + \underbrace{\left(\frac{d}{dx}y\right)}_{\frac{dy}{dx}} x$$

$$= y + x\frac{dy}{dx}$$

$$\left(y + x\frac{dy}{dx}\right)(-\sin(xy)) = \left(y + x\frac{dy}{dx}\right)\cos(xy)$$

$$-y\sin(xy) - x\sin(xy)\frac{dy}{dx} = y\cos(xy) + x\cos(xy)\frac{dy}{dx}$$

$$-y(\sin(xy)+\cos(xy)) = \underbrace{x(\sin(xy)+\cos(xy))}\frac{dy}{dx}$$

Same ... so divide off

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(x,y)=\left(\frac{\sqrt{\pi}}{2},\frac{\sqrt{\pi}}{2}\right)} = -\frac{\sqrt{\pi}/2}{\sqrt{\pi}/2} = -1$$

$\Rightarrow$ tan line is

$$y - \frac{\sqrt{\pi}}{2} = -1\left(x - \frac{\sqrt{\pi}}{2}\right)$$

NEXT

How to take derivatives of arcsin + friends.