Chain Rule

Two ways to think about it:

1. Leibniz notation

$$\frac{dy}{dx} = \frac{d\xi}{dx}\frac{dy}{d\xi}$$

2. function composition notation

$$\frac{d}{dx}\underbrace{f(g(x))} = f'(g(x))g'(x)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

match

$$\frac{d}{dx}e^x = e^x$$

Ex: $y = \sin(e^x)$

Compute $\frac{dy}{dx}$.

Soln: $\frac{dy}{dx} = \frac{d}{dx}\left[\sin(e^x)\right]$

do not match $\Rightarrow$ use chain rule!

$$= \frac{d(e^x)}{dx}\frac{d}{de^x}\left[\sin(e^x)\right]$$

match

$$= e^x\cos(e^x)$$

Ex: $\frac{d}{d\xi} e^{5\xi}$ mismatch $= \frac{d(5\xi)}{d\xi} \frac{d}{d(5\xi)} e^{5\xi}$ match!

$= 5e^{5\xi}$

$e^{5x} = f(g(x))$

where $f(x) = e^x$ $g(x) = 5x$

$\ln(a^b) = b\ln(a)$

Ex: $\frac{d}{dx}(3^x)$

Soln: Algebra

Recall $e^{\ln(x)} = x$

$3^x = e^{\ln(3^x)}$

$= e^{x\ln(3)}$

So, $\frac{d}{dx} 3^x = \frac{d}{dx} e^{x\ln(3)}$ mismatch $= \frac{d(x\ln(3))}{dx} \frac{d}{d(x\ln(3))} e^{x\ln(3)}$ match!

by ALG

$= \ln(3) e^{x\ln(3)}$

$= (3^x)\ln(3)$

Ex: $\frac{d}{d\alpha}\sin(\cos(e^{\alpha}))$

mismatch!

$$\frac{d}{d\alpha}\sin(\cos(e^{\alpha})) = \frac{d(\cos(e^{\alpha}))}{d\alpha}\underbrace{\frac{d}{d(\cos(e^{\alpha}))}\sin(\cos(e^{\alpha}))}_{}$$

$\cos(\cos(e^{\alpha}))$

mismatch! $\Downarrow$ another chain rule

match!

$$= \left[\frac{d(e^{\alpha})}{d\alpha}\right]\left[\frac{d}{d(e^{\alpha})}\cos(e^{\alpha})\right]\cos(\cos(e^{\alpha}))$$

match ✓ match ✓

$$= e^{\alpha}(-\sin(e^{\alpha}))\cos(\cos(e^{\alpha}))$$

Ex: $\frac{d}{d\beta}2^{(3^{\beta})}$

Soln: By algebra

$$(\star) \quad 2^{(3^{\beta})} = e^{\ln(2^{(3^{\beta})})}$$

$$= e^{3^{\beta}\ln(2)}$$

$$3^{\beta} = e^{\ln(3^{\beta})} = e^{\beta\ln(3)}$$

So in $(\star)$ we actually have

$$2^{(3^{\beta})} = e^{(e^{\beta\ln(3)})\ln(2)}$$

So,

$$\frac{d}{d\beta} 2^{(3^\beta)} = \frac{d}{d\beta} e^{\left(e^{\beta \ln(3)}\right)\ln(2)}$$ mismatch

$$= \frac{d\left(\left(e^{\beta\ln(3)}\right)\boxed{\ln(2)}\right)}{d\beta} \frac{d}{d\left(\ln(2) e^{\beta \ln(3)}\right)} e^{\left(e^{\beta\ln(3)}\right)\ln(2)}$$ mismatch, match!

$$= \ln(2)\left[\frac{d\left(\beta \boxed{\ln(3)}\right)}{d\beta} \frac{d}{d(\beta\ln(3))} e^{\beta\ln(3)}\right] e^{\left(e^{\beta\ln(3)}\right)\ln(2)}$$ match

$$\underbrace{e^{\left(e^{\beta\ln(3)}\right)\ln(2)}}_{=2^{(3^\beta)}}$$

$$= \ln(2)\left[\ln(3)(1)e^{\beta\ln(3)}\right]2^{(3^\beta)}$$

$$= \ln(2)\ln(3)\, 3^\beta\, 2^{(3^\beta)}$$

<u>Ex</u>: $\frac{d}{d\psi} \ln(\ln(\ln(\psi)))$ mismatch

$$\frac{d}{d\xi}\ln(\xi) = \frac{1}{\xi}$$

$$= \frac{d(\ln(\ln(\psi)))}{d\psi} \frac{d}{d[\ln(\ln(\psi))]} \ln(\ln(\ln(\psi)))$$ mismatch, match

$$= \frac{d(\ln(\psi))}{d\psi}\left[\frac{d}{d(\ln(\psi))}\ln(\ln(\psi))\right]\cdot\frac{1}{\ln(\ln(\psi))}$$ match

$$= \frac{1}{\psi}\frac{1}{\ln(\psi)}\cdot\frac{1}{\ln(\ln(\psi))}$$

Ex: tan line to $y = \ln(x^3+2x^2+1)$

at $x=2$.

Soln: at $x=2$: $y = \ln(2^3+2(2^2)+1)$

$= \ln(8+8+1)$

$= \ln(17)$

$\Rightarrow (2, \ln(17))$ lies on graph

$$\frac{dy}{dx} = \frac{d}{dx} \ln(x^3+2x^2+1)$$

mismatch

$$= \frac{d(x^3+2x^2+1)}{dx} \frac{d}{d(x^3+2x^2+1)} \ln(x^3+2x^2+1)$$

$$= (3x^2+4x)\frac{1}{x^3+2x^2+1}$$

at $x=2$: $\left.\frac{dy}{dx}\right|_{x=2} = \frac{3(2^2)+4(2)}{2^3+2(2^2)+1}$

$$= \frac{3(4)+8}{8+8+1} = \frac{20}{17}$$

tan line: $y - \ln(17) = \frac{20}{17}(x-2)$

So far.... only looked at functions.
What if we considered a graph that isn't a function?

Ex: Let's find a tan line to Cissoid of Diocles

$$(x^2+y^2)x = 2y^2$$

at $(1,1)$.

Soln: We need a slope $\sim \frac{dy}{dx}$.

Take $\frac{d}{dx}$ of both sides: $\quad \frac{d}{dx} x^n = nx^{n-1}$

$$\frac{d}{dx}\underbrace{\left[(x^2+y^2)x\right]}_{\text{prod rule}} = \frac{d}{dx}\left[2y^2\right]$$

$$\underbrace{\frac{d}{dx}[x^2+y^2]}\,x + (x^2+y^2)\underbrace{\frac{d}{dx}[x]}_{=1} = 2\frac{d}{dx}[y^2]$$

$$\left[2x + \frac{d}{dx}y^2\right]x + x^2+y^2 = 2\frac{d}{dx}[y^2]$$

$$\left[2x + \frac{dy}{dx}\frac{d}{dy}y^2\right]x + x^2+y^2 = 2\frac{dy}{dx}\frac{d}{dy}[y^2]$$

slope

$$\left[2x+\boxed{\frac{dy}{dx}}\cdot 2y\right]x + x^2+y^2 = 2\boxed{\frac{dy}{dx}}(2y)$$

$$\frac{dy}{dx}\left[2yx-4y\right] = -x^2-y^2-2x^2$$

$$\frac{dy}{dx} = \frac{-3x^2-y^2}{2yx-4y}$$

So slope:

$$\left.\frac{dy}{dx}\right|_{x=1,y=1} = \frac{-3-1}{2-4} = \frac{-4}{-2} = 2$$

$\Rightarrow$ tan line is

$$y-1 = 2(x-1)$$