Ex: Solve $\sin(3\theta) = -\frac{1}{2}$ for θ in $[0, 2\pi]$.

Soln: Introduce a new variable: $\psi = 3\theta$

$2\pi = \frac{36\pi}{18}$

$\sin(\psi) = -\frac{1}{2}$

Unit circle: fundamental solns $\sim \psi = \frac{7\pi}{6}, \frac{11\pi}{6}$

General soln: for $k \in \mathbb{Z}$,

$3\theta = \psi = \frac{7\pi}{6} + 2\pi k = \frac{7\pi + 12\pi k}{6}$

$3\theta = \psi = \frac{11\pi}{6} + 2\pi k = \frac{11\pi + 12\pi k}{6}$

$\Rightarrow \theta = \frac{7\pi + 12\pi k}{18}$

$\theta = \frac{11\pi + 12\pi k}{18}$

find all integers k such that $0 \le \theta \le \frac{36\pi}{18}$

$k = 3 \to \theta = \frac{7\pi + 36\pi}{18}$ ✗ too big

rest too big

$k = 2 \to \theta = \frac{31\pi}{18}$ ✓

$k = 1 \to \theta = \frac{19\pi}{18}$ ✓

$k = 0 \to \theta = \frac{7\pi}{18}$ ✓

$k = -1 \to \theta = -\frac{5\pi}{18}$ ✗

$k = -2$ ↓ too negative

$k = 3 \to \theta = \frac{11\pi + 36\pi}{18}$ ✗ too big

$k = 2 \to \theta = \frac{35\pi}{18}$ ✓

$k = 1 \to \theta = \frac{23\pi}{18}$ ✓

$k = 0 \to \theta = \frac{11\pi}{18}$ ✓

$k = -1 \to \theta = \frac{11\pi - 12\pi}{18} = -\frac{\pi}{18}$ ✗

↓ too negative

We see there are six solutions $\sim \theta = \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$

Ex: Solve $\cos(\pi\theta) = \frac{\sqrt{2}}{2}$ for θ in $[0, 2\pi]$

Soln: Introduce new variable

$$\psi = \pi\theta$$

Equation becomes

$$\cos(\psi) = \frac{\sqrt{2}}{2}$$

Fundamental soln: $\frac{\pi}{4}, \frac{7\pi}{4}$

$$\frac{\frac{\pi}{4}}{\pi} = \frac{\pi/4}{\pi/1} = \frac{\pi}{4} \cdot \frac{1}{\pi}$$

General soln: for $k \in \mathbb{Z}$,

$$\pi\theta = \psi = \frac{\pi}{4} + 2\pi k$$

$$\pi\theta = \psi = \frac{7\pi}{4} + 2\pi k$$

Divide by π

$$\theta = \frac{\frac{\pi}{4} + 2\pi k}{\pi} = \frac{\frac{\pi}{4}}{\pi} + \frac{2\pi k}{\pi}$$

$$= \frac{1}{4} + 2k$$

$$= \frac{1 + 8k}{4}$$

$0 \le \theta \le 2\pi$

$0 \le \theta \le 6.28$

too big

$k = 4 \rightarrow \theta = \frac{33}{4} = 8.25$ X

$k = 3 \rightarrow \theta = \frac{25}{4} = 6.25$ ✓

$k = 2 \rightarrow \theta = 17/4 = 4.25$ ✓

$k = 1 \rightarrow \theta = 9/4 = 2.25$ ✓

$k = 0 \rightarrow \theta = 1/4 = 0.25$ ✓

$k = -1 \rightarrow \theta = -1/4 < 0$ X

$$\theta = \frac{7}{4} + 2k = \frac{7 + 8k}{4}$$

don't work

$k = 3$ $\theta = 31/4 = 7.75$ X

$k = 2$ $\theta = 23/4 = 5.75$ ✓

$k = 1$ $\theta = \frac{15}{4} = 3.75$ ✓

$k = 0$ $\theta = \frac{7}{4} = 1.75$ ✓

$k = -1$ $\theta = -\frac{1}{4} = -0.25$ X

no work

So we get seven solns:

$\theta = \frac{1}{4}, \frac{7}{4}, \frac{9}{4}, \frac{15}{4}, \frac{17}{4}, \frac{23}{4}, \frac{25}{4}$