Sum + difference identities

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \longrightarrow \frac{d}{dx}\sin x$$

$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\rightarrow \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

Ex: $$\cos\left(\underset{\substack{\uparrow \\ a=x}}{x} + \underset{\substack{\uparrow \\ b=\frac{3\pi}{4}}}{\frac{3\pi}{4}}\right) = \underline{\cos\left(\frac{3\pi}{4}\right)}\ \underset{\substack{\uparrow \\ \text{"}\cos(a)\text{"}}}{\cos(x)} - \underline{\sin\left(\frac{3\pi}{4}\right)}\ \underset{\substack{\uparrow \\ \text{"}\sin(a)\text{"}}}{\sin(x)}$$

$$= -\frac{\sqrt{2}}{2}\cos(x) - \frac{\sqrt{2}}{2}\sin(x)$$

Ex: $$\sin(2\theta) = \sin\left(\overset{\substack{a=\theta \\ \downarrow}}{\theta} + \overset{\substack{b=\theta \\ \downarrow}}{\theta}\right)$$

$$= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)$$

$$= 2\sin(\theta)\cos(\theta)$$

Ex: $$\sin(3\theta) = \sin\left(\underset{\substack{\uparrow \\ a=\theta}}{\theta} + \underset{\substack{\uparrow \\ b=2\theta}}{2\theta}\right) = \sin(\theta)\overbrace{\cos(2\theta)}^{=(\cos^2(\theta)-\sin^2(\theta))} + \cos(\theta)\overbrace{\sin(2\theta)}^{=2\sin(\theta)\cos(\theta)}$$

$$= \sin(\theta)\cos^2(\theta) - \sin^3(\theta) + 2\sin(\theta)\cos(\theta)$$

Ex: (similar to WHW 13)

Simplify

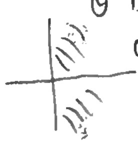
$$\sin(\arctan(x))$$
$$\sin(2\arctan(x))$$
$$\sin(3\arctan(x))$$

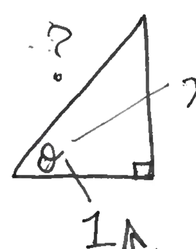
Soln:

$\underline{\sin(\arctan(x))}$

Let $\theta = \arctan(x) \longrightarrow \tan(\theta) = x = \dfrac{x}{1}$.

θ is QI or QIV





$1^2 + x^2 = ?^2$

$? = +\sqrt{1+x^2}$

Therefore,

$$\boxed{\sin(\arctan(x)) = \sin(\theta) = \frac{x}{\sqrt{1+x^2}}}$$

$\underline{\sin(2\arctan(x))}$

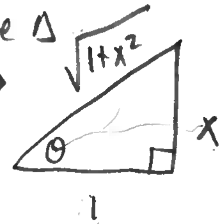
Let $\theta = \arctan x$

double angle identity

$$\sin(2\arctan x) = \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$= 2\left(\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{\sqrt{1+x^2}}\right)$$
$$= \frac{2x}{1+x^2}$$

$\sin(3\arctan(x))$

Let $\theta = \arctan(x)$. $\longrightarrow$ Same Δ



Then

$$\sin(3\arctan(x)) = \sin(3\theta)$$

$$= \sin(\theta + 2\theta)$$

$$a = \theta \qquad b = 2\theta$$

$$= \sin(\theta)\cos(2\theta) + \cos(\theta)\sin(2\theta)$$

previous calculation

Using double angle identity

$$= \sin(\theta)\left[\cos^2(\theta) - \sin^2(\theta)\right] + \cos(\theta)\boxed{\left[\frac{2x}{1+x^2}\right]}$$

$$= \frac{x}{\sqrt{1+x^2}}\left[\left(\frac{1}{\sqrt{1+x^2}}\right)^2 - \left(\frac{x}{\sqrt{1+x^2}}\right)^2\right] + \frac{1}{\sqrt{1+x^2}}\left(\frac{2x}{1+x^2}\right)$$

$$= \frac{1}{\sqrt{1+x^2}}\left\{\left[\frac{x}{1+x^2} - \frac{x^3}{1+x^2}\right] + \frac{2x}{1+x^2}\right]$$

$$= \frac{1}{\sqrt{1+x^2}}\left[\frac{1}{1+x^2}\right]\left[3x - x^3\right]$$

Ex: $\tan\left(\frac{\pi}{6} - \frac{3\pi}{4}\right) = \dfrac{\sin\left(\frac{\pi}{6} - \frac{3\pi}{4}\right)}{\cos\left(\frac{\pi}{6} - \frac{3\pi}{4}\right)}$

$\frac{2\pi}{12} - \frac{9\pi}{12}$

$= -\frac{7\pi}{12}$

$a = \frac{\pi}{6}$
$b = \frac{3\pi}{4}$

$$= \frac{\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{3\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{\pi}{6}\right)\sin\left(\frac{3\pi}{4}\right)}$$

$$= \frac{\left(\frac{1}{\cancel{2}}\right)\left(-\frac{\sqrt{2}}{\cancel{2}}\right) - \left(\frac{\sqrt{3}}{\cancel{2}}\right)\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right)$$

$$= \frac{\left(\frac{-\sqrt{2} - \sqrt{6}}{4}\right)}{\left(\frac{-\sqrt{6} + \sqrt{2}}{4}\right)}$$

$$= \left(\frac{-\sqrt{2} - \sqrt{6}}{\cancel{4}}\right)\left(\frac{\cancel{4}}{-\sqrt{6} + \sqrt{2}}\right)$$

$$= \frac{-\sqrt{2} - \sqrt{6}}{-\sqrt{6} + \sqrt{2}}$$

↑

DO NOT write as decimal

Ex: Given $\sin(u) = \frac{1}{10}$ and $\cos(u)$ is negative.

u is in QI or QII

u is in QII or QIII

Compute

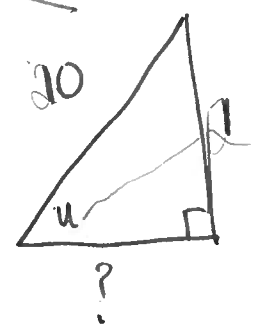
$\cos(u) =$

$\sin(u-\pi) =$

$\cos(u-\pi) =$

$\sin(u-\frac{\pi}{2}) =$

$\cos(u-\frac{\pi}{2}) =$



$?^2 + 1^2 = 10^2$

$? = \sqrt{99}$

Soln:

$\cos(u) = -\frac{\sqrt{99}}{10}$

$\sin(u-\pi) = \sin(u)\cos(\pi) - \cos(u)\sin(\pi)$

$= (-1)(\frac{1}{10}) - \cancel{(-\frac{\sqrt{99}}{10})(0)} = -\frac{1}{10}$

$\cos(u-\pi) = \cos(u)\cos(\pi) + \sin(u)\sin(\pi)$

$= (-\frac{\sqrt{99}}{10})(-1) + (\frac{1}{10})(0)$

$= \frac{\sqrt{99}}{10}$

$\sin(u-\frac{\pi}{2}) = \cancel{\sin(u)\underbrace{\cos(\frac{\pi}{2})}_{=0}} - \cos(u)\overbrace{\sin(\frac{\pi}{2})}^{=1} = \frac{\sqrt{99}}{10}$

$\cos(u-\frac{\pi}{2}) = \cancel{\cos(u)\underbrace{\cos(\frac{\pi}{2})}_{=0}} + \sin(u)\sin(\frac{\pi}{2}) = \frac{1}{10}$