Half-angle identities

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$\sin>0$, $\cos<0$, $\tan<0$	$\sin>0$, $\cos>0$, $\tan>0$
$\sin<0$, $\cos<0$, $\tan>0$	$\sin<0$, $\cos>0$, $\tan<0$

in QII: $\frac{\pi}{2} \le \theta \le \pi$

QI: $\frac{\pi}{4} \le \frac{\theta}{2} \le \frac{\pi}{2}$

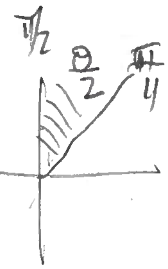
QIII: $\pi \le \theta \le \frac{3\pi}{2}$

QII: $\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{3\pi}{4}$

Ex: If $\tan(\theta) = \frac{-1}{4}$ and $\cos(\theta) < 0$, compute

$\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.

Soln: $\underbrace{\tan(\theta) = -\frac{1}{4}}_{\theta \text{ is in QII or QIV}}$ $\qquad \underbrace{\cos(\theta) < 0}_{\theta \text{ is QII or QIII}}$

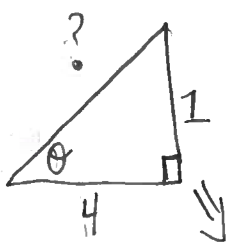


$\Downarrow$

θ is in QII

$\Downarrow$

$\frac{\theta}{2}$ is in QI $\Rightarrow$ $\boxed{\sin\left(\frac{\theta}{2}\right) > 0 \quad \cos\left(\frac{\theta}{2}\right) > 0}$



$\Downarrow$

$4^2 + 1^2 = ?^2$

$? = \pm\sqrt{17}$

So,

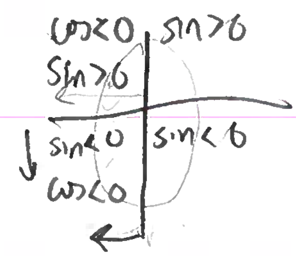
$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}} = \sqrt{\frac{1-\frac{4}{\sqrt{17}}}{2}}$$

$$= \sqrt{\frac{1}{2} - \frac{2}{\sqrt{17}}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}} = \sqrt{\frac{1+\frac{4}{\sqrt{17}}}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{\sqrt{\frac{1-4/\sqrt{17}}{2}}}{\sqrt{\frac{1+4/\sqrt{17}}{2}}} = \sqrt{\frac{1-4/\sqrt{17}}{1+4/\sqrt{17}}}$$

Ex: If $\sin(\theta) = -\frac{2}{11}$ and $\cos(\theta) < 0$, compute $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, AND $\tan\left(\frac{\theta}{2}\right)$. ②.



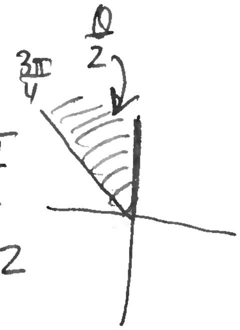
Sol'n: $\underbrace{\sin(\theta) = -\frac{2}{11}}_{\theta \text{ is in QIII or QIV}}$ $\qquad$ $\underbrace{\cos(\theta) < 0}_{\theta \text{ is in QII or QIII}}$

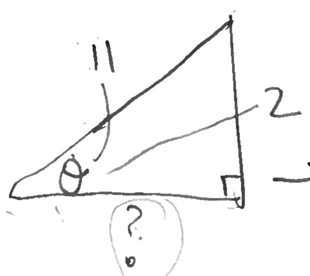
$\boxed{\theta \text{ is in QIII}} \Longrightarrow \pi \le \theta \le \frac{3\pi}{2}$

$\Downarrow$ div by 2

$\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{3\pi}{4}$

$\Rightarrow \boxed{\frac{\theta}{2} \text{ is in QII}}$





$\rightarrow ?^2 + 2^2 = 11^2$

$? = \sqrt{121-4} = \sqrt{117}$

So, compute

$$\sin\left(\frac{\theta}{2}\right) = +\sqrt{\frac{1-\cos(\theta)}{2}} = +\sqrt{\frac{1-\left(-\frac{\sqrt{117}}{11}\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{117}}{11}}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1+\cos(\theta)}{2}} = -\sqrt{\frac{1-\frac{\sqrt{117}}{11}}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \frac{\sqrt{\frac{1+\frac{\sqrt{117}}{11}}{2}}}{-\sqrt{\frac{1-\frac{\sqrt{117}}{11}}{2}}} = -\sqrt{\frac{1+\frac{\sqrt{117}}{11}}{1-\frac{\sqrt{117}}{11}}} = -\sqrt{\frac{11+\sqrt{117}}{11-\sqrt{117}}}$$

Ex: Compute $\sin\left(\frac{\pi}{8}\right)$.

not on unit circle

$\uparrow$ interpret using half angle identity

Soln: Write $\boxed{\frac{\pi}{8} = \frac{\theta}{2}} \xrightarrow{\text{mult by } 2} \frac{2\pi}{8} = \theta \longrightarrow \boxed{\theta = \frac{\pi}{4}}$

$\frac{\theta}{2}$ is in QI

θ is in QI

Compute

$$\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{\theta}{2}\right) = \sin\left(\frac{\pi/4}{2}\right)$$

$$= +\sqrt{\frac{1-\cos(\pi/4)}{2}}$$

$$= \sqrt{\frac{1-\sqrt{2}/2}{2}}$$

Ex: Compute $\cos\left(\frac{\pi}{16}\right)$.

not on unit circle (is in QI)

Soln: Write $\frac{\pi}{16} = \frac{\theta}{2} \longrightarrow \theta = \frac{\pi}{8}$.

Compute $\boxed{\cos\left(\frac{\pi}{16}\right) = \cos\left(\frac{\theta}{2}\right) = +\sqrt{\frac{1+\cos\left(\frac{\pi}{8}\right)}{2}}}$ $(\star)$

Write $\frac{\pi}{8} = \frac{\psi}{2} \longrightarrow \psi = \frac{\pi}{4} \Rightarrow \cos\left(\frac{\pi}{8}\right) = +\sqrt{\frac{1+\cos(\pi/4)}{2}}$

QI

$= \cos\left(\frac{\psi}{2}\right) = \sqrt{\frac{1+\sqrt{2}/2}{2}}$

So by $(\star)$: $\cos\left(\frac{\pi}{16}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{8}\right)}{2}} = \sqrt{\frac{1+\sqrt{\frac{1+\sqrt{2}/2}{2}}}{2}}$

Ex: Simplify

$$\tan\Big(2\,\underbrace{\arcsin\left(\tfrac{x}{4}\right)}_{\theta}\Big)$$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$\boxed{= 1 - 2\sin^2(\theta)}$

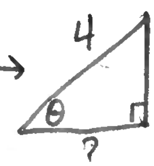
$= 2\cos^2(\theta) - 1$

Sol'n: $$\tan\left(2\arcsin\left(\tfrac{x}{4}\right)\right) = \frac{\sin\left(2\arcsin\left(\frac{x}{4}\right)\right)}{\cos\left(2\arcsin\left(\frac{x}{4}\right)\right)}$$

$$= \frac{2\cancel{\sin}\left(\cancel{\arcsin}\left(\frac{x}{4}\right)\right)\cos\left(\arcsin\left(\frac{x}{4}\right)\right)}{1 - 2\cancel{\sin}^2\left(\cancel{\arcsin}\left(\frac{x}{4}\right)\right)}$$

$$= \frac{2\left(\frac{x}{4}\right)\cos\left(\arcsin\left(\frac{x}{4}\right)\right)}{1 - 2\left(\frac{x}{4}\right)^2}$$

Let $\theta = \arcsin\left(\frac{x}{4}\right) \rightarrow \sin\theta = \frac{x}{4} \longrightarrow$



$\rightarrow ?^2 + x^2 = 16$

$? = \sqrt{16 - x^2}$

Therefore,

$$\cos\left(\arcsin\left(\tfrac{x}{4}\right)\right) = \cos(\theta) = \frac{\sqrt{16 - x^2}}{4}$$

Thus,

$$\tan\left(2\arcsin\left(\tfrac{x}{4}\right)\right) = \frac{\frac{x}{2}\cos\left(\arcsin\left(\frac{x}{4}\right)\right)}{1 - \frac{x^2}{8}}$$

$$= \frac{\frac{x}{2}\,\frac{\sqrt{16 - x^2}}{4}}{1 - \frac{x^2}{8}} = \frac{x\sqrt{16 - x^2}}{8 - x^2}$$