Quiz 4 MATH 3520 Fall 2019

Prove that if $\mathcal{B}$ is a basis for a vector space V and $\vec{u} \in V$, then for any scaler $\alpha \in \mathbb{R}$, $[\alpha\vec{u}]_{\mathcal{B}} = \alpha[\vec{u}]_{\mathcal{B}}$.

Proof: Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V, and let $\vec{u} \in V$.

Since $\mathcal{B}$ is a basis, there are (unique) scalars $\alpha_1, \ldots, \alpha_n$ so that

$$\vec{u} = \alpha_1\vec{v}_1 + \ldots + \alpha_n\vec{v}_n,$$

and hence

$$(\ast) \qquad [\vec{u}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

If $\alpha \in \mathbb{R}$, we may compute

$$\alpha\vec{u} = \alpha(\alpha_1\vec{v}_1 + \ldots + \alpha_n\vec{v}_n) = (\alpha\alpha_1)\vec{v}_1 + \ldots + (\alpha\alpha_n)\vec{v}_n$$

and so we see

$$[\alpha\vec{u}]_{\mathcal{B}} = \begin{bmatrix} \alpha\alpha_1 \\ \vdots \\ \alpha\alpha_n \end{bmatrix}$$

On the other hand, from $(\ast)$ we see

$$\alpha[\vec{u}]_{\mathcal{B}} = \alpha\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha\alpha_1 \\ \vdots \\ \alpha\alpha_n \end{bmatrix} = [\alpha\vec{u}]_{\mathcal{B}},$$

completing the proof. $\blacksquare$