HW 9 MATH 3520 Fall 2019

§6.5 #2 $\begin{cases} T: \mathbb{R}^{2\times 2} \to \mathbb{R} \\ T(A) = \mathrm{tr}(A) \end{cases}$

(a) (i) $T\left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\right) = 1+3 = 4$ ~ so $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \notin \ker(T)$

(ii) $T\left(\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}\right) = 0+0 = 0$ — so $\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} \in \ker(T)$

(iii) $T\left(\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}\right) = 1+(-1) = 0$ ~ so $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \in \ker(T)$

(b) all scalers are in range(T)

(c) find ker(T)

$$0 = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+d$$

$$\Downarrow$$

$$a = -d$$

Therefore

$$\ker(T) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} ; a, b, c \in \mathbb{R} \right\}$$

find range(T)

Since for any $\alpha \in \mathbb{R}$, $T\left(\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}\right) = \alpha$,

any $\alpha \in \mathbb{R}$ is in range(T), i.e.

$$\mathrm{range}(T) = \mathbb{R}$$

#4 $\begin{cases} T: \mathcal{P}_2 \to \mathcal{P}_2 \\ T(p) = xp'(x) \end{cases}$

a) (i) $T(1) = x\frac{d}{dx}(1) = 0 \rightarrow 1 \in \ker(T)$

(ii) $T(x) = x\frac{d}{dx}(x) = x(1) = x \rightarrow x \notin \ker(T)$

(iii) $T(x^2) = x\frac{d}{dx}(x^2) = 2x^2 \longrightarrow x^2 \notin \ker(T)$

b) $x \in \text{Range}(T)$ because $T(x) = x\frac{d}{dx}(x) = x$

and

$x^2 \in \text{Range}(T)$ because $T\left(\frac{x^2}{2}\right) = x\frac{d}{dx}\left(\frac{x^2}{2}\right) = x^2$

but

$1 \notin \text{Range}(T)$ because for it to be there $xp'(x) = 1$

$\downarrow$

$p'(x) = \frac{1}{x}$

$\downarrow$

$p(x) = \ln(x) + C$

not in $\mathcal{P}$

c) Ker(T)

$T(ax^2+bx+c) = x\frac{d}{dx}(ax^2+bx+c) = 2ax^2+bx \overset{\uparrow}{=} 0$

be in Ker(T)

$\Downarrow$

$2a = 0 \rightarrow a = 0$

$b = 0 \rightarrow b = 0$

no Restriction on c ← free var

Therefore

$\ker(T) = \{c : c \in \mathbb{R}\}$

range(T) Let $\alpha x^2 + \beta x + \gamma \in \mathcal{P}_2$ be arbitrary and consider

$\alpha x^2 + \beta x + \gamma = T(ax^2+bx+c) = 2ax^2 + bx$

$\Rightarrow$ $2a = \alpha \rightarrow a = \frac{\alpha}{2}$

$b = \beta$

$\gamma = 0$

Therefore,

$\text{range}(T) = \{ax^2 + bx : a, b \in \mathbb{R}\}$

#9) $\begin{cases} T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 1} \\ T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b \\ c-d \end{bmatrix} \end{cases}$

Soln: First find Ker(T):

$$\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix} =}_{\text{in Kernel}} T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \underbrace{=}_{\text{calculated}} \begin{bmatrix} a-b \\ c-d \end{bmatrix}$$

$$\Rightarrow \begin{cases} a-b=0 \\ c-d=0 \end{cases} \to \begin{cases} a=b \\ c=d \end{cases}$$

Therefore,

$$\operatorname{Ker}(T) = \left\{ \begin{bmatrix} a & a \\ c & c \end{bmatrix} : a, c \in \mathbb{R} \right\}$$

and ker(T) has a basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

and hence

$$\operatorname{nullity}(T) = \dim \ker(T) = 2$$

Since $\dim \mathbb{R}^{2\times 2} = 4$, the Rank-Nullity theorem shows

$$\operatorname{rank}(T) + 2 = 4,$$

hence we conclude

$$\operatorname{rank}(T) = 2.$$

#10) $\begin{cases} T: \mathcal{P}_2 \to \mathbb{R}^{2\times 2} \\ T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} \end{cases}$

Soln: Find $\ker(T)$: let $ax^2+bx+c \in \mathcal{P}$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{to be in} \\ \text{kernel}}}{=} T(ax^2+bx+c) \underset{\substack{\uparrow \\ \text{calculated}}}{=} \begin{bmatrix} c \\ a+b+c \end{bmatrix}$$

$$\Rightarrow \begin{cases} c = 0 \\ a+b+0 = 0 \to a = -b \end{cases}$$

Therefore,

$$\ker(T) = \{ax^2 - ax : a \in \mathbb{R}\}$$

Thus, $\ker(T)$ has basis $\mathcal{B} = \{x^2 - x\}$

$$\text{nullity}(T) = \dim \ker(T) = 1$$

Since $\dim \mathcal{P}_2 = 3$, the rank-nullity theorem says

$$3 = \text{rank}(T) + 1$$

$$\Downarrow$$

$$\text{rank}(T) = 2$$

#13) $\begin{cases} T: \mathcal{P}_2 \to \mathbb{R} \\ T(p(x)) = p'(0) \end{cases}$

Soln: Find the kernel of T:

$$0 \underset{\substack{\uparrow \\ \text{in the} \\ \text{kernel}}}{=} T(ax^2+bx+c) \underset{\substack{\uparrow \\ \text{calculate}}}{=} 2a(0)+b = b$$

$$\Rightarrow \quad 0 = b$$

Therefore,

$$\ker(T) = \{ax^2+c : a, c \in \mathbb{R}\}$$

which has basis $\mathcal{B} = \{x^2, 1\}$. Therefore, nullity$(T) = \dim \ker(T) = 2$. Since $\dim \mathcal{P}_2 = 3$, the rank-nullity theorem implies

$$3 = 2 + \text{rank}(T),$$

hence

$$\text{rank}(T) = 1.$$

Problem A $\begin{cases} T_n : \mathcal{P}_2 \to \mathcal{P}_2 \\ T_n(p) = (1-x^2)p'' - xp' + n^2 p \end{cases}$

n=0

$$\begin{cases} T_0 : \mathcal{P}_2 \to \mathcal{P}_2 \\ T_0(p) = (1-x^2)p'' - xp' \end{cases}$$

Find Ker(T_0)

$$0 = T(ax^2+bx+c) = (1-x^2)(2a) - x(2ax+b)$$

$$= 2a - 2ax^2 - 2ax^2 - 2bx$$

$$= (-4a)x^2 + (-b)x + 2a$$

$$-4a = 0 \to a = 0$$
$$-b = 0 \to b = 0$$
$$2a = 0 \to a = 0$$

$$\Rightarrow \ker(T_0) = \{c : c \in \mathbb{R}\}$$

n=1

$$\begin{cases} T_1 : \mathcal{P}_2 \to \mathcal{P}_2 \\ T_1(p) = (1-x^2)p'' - xp' + p \end{cases}$$

Find Ker(T_1)

$$0 = T_1(ax^2+bx+c) = (1-x^2)(2a) - x(2ax+b) + (ax^2+bx+c)$$

$$= (-2a-2a+a)x^2 + (-b+b)x + (2a+c)$$

$$= -3ax^2 + (2a+c)$$

$$\Rightarrow \begin{cases} -3a = 0 \to a = 0 \\ 2a+c = 0 \to c = 0 \end{cases}$$

$$\Rightarrow \ker(T_1) = \{bx : b \in \mathbb{R}\}$$

<u>$n=2$</u>

$$\begin{cases} T_2 : \mathcal{P}_2 \to \mathcal{P}_2 \\ T_2(p) = (1-x^2)p'' - xp' + 4p \end{cases}$$

<u>Find $\ker(T_2)$</u>

$$0 = T_2(ax^2+bx+c) = (1-x^2)(2a) - x(2ax+b) + 4(ax^2+bx+c)$$

$$= (\underbrace{-2a-2a+4a}_{0})x^2 + (-b+4b)x + (2a+4c)$$

$$= 3bx + (2a+4c)$$

$$\Rightarrow \begin{cases} 3b=0 \to b=0 \\ 2a+4c=0 \to a=2c. \end{cases}$$

$$\Rightarrow \ker(T_2) = \{2cx^2+c;\ c \in \mathbb{R}\}$$