Homework 7 MATH 3520 Fall 2019

§6.3 #2 $\vec{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $\mathcal{C} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

a) Find $[x]_{\mathcal{B}}$

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & -1 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right]$$

$$\downarrow$$

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\downarrow$$

$$[x]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Find $[x]_{\mathcal{C}}$

$$\alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 0 & 2 & 4 \\ 1 & 3 & -1 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 2 \end{array}\right]$$

$$\downarrow$$

$$[x]_{\mathcal{C}} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$$

b) $P_{\mathcal{C} \leftarrow \mathcal{B}} \overset{\text{def on p. 465}}{=} \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{C}} \; \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}} \right]$

Find $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{C}}$

$$\alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 2 & 1 \\ 1 & 3 & 0 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \end{array}\right]$$

$$\downarrow$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$$

Find $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}}$

$$\alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 0 & 2 & 1 \\ 1 & 3 & 1 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{array}\right]$$

$$\downarrow$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

Therefore,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -3/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}$$

c) $[x]_{\mathcal{C}} \stackrel{?}{=} P_{\mathcal{C}\leftarrow\mathcal{B}}[x]_{\mathcal{B}}$

$$\begin{bmatrix}-7\\2\end{bmatrix} \stackrel{?}{=} \frac{1}{2}\begin{bmatrix}-3 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}5\\-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-15+1\\5-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}-14\\4\end{bmatrix}$$

$$= \begin{bmatrix}-7\\2\end{bmatrix}$$

yes!

using $\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

d) $$P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix}-3/2 & -1/2\\1/2 & 1/2\end{bmatrix}^{-1} = \frac{1}{-\frac{3}{4}+\frac{1}{4}}\begin{bmatrix}1/2 & 1/2\\-1/2 & -3/2\end{bmatrix}$$

$$= -2\begin{bmatrix}1/2 & 1/2\\-1/2 & -3/2\end{bmatrix}$$

$$= \begin{bmatrix}-1 & -1\\1 & 3\end{bmatrix}$$

e) $[x]_{\mathcal{B}} \stackrel{?}{=} P_{\mathcal{B}\leftarrow\mathcal{C}}[x]_{\mathcal{C}}$

$$\begin{bmatrix}5\\-1\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}-1 & -1\\1 & 3\end{bmatrix}\begin{bmatrix}-7\\2\end{bmatrix}$$

$$= \begin{bmatrix}7-2\\-7+6\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix}$$

yes!

§6.4

#1 Linear:

1

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = T\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right)$$

$$= \begin{bmatrix} a+e+b+f & 0 \\ 0 & c+g+d+h \end{bmatrix}$$

equal!!

while

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix} + \begin{bmatrix} e+f & 0 \\ 0 & g+h \end{bmatrix}$$

$$= \begin{bmatrix} a+b+e+f & 0 \\ 0 & c+d+g+h \end{bmatrix}$$

2

$$T\left(\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}\right) = \begin{bmatrix} \alpha a+\alpha b & 0 \\ 0 & \alpha c+\alpha d \end{bmatrix}$$

$\updownarrow$ equal!!

while

$$\alpha T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \alpha\begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix} = \begin{bmatrix} \alpha(a+b) & 0 \\ 0 & \alpha(c+d) \end{bmatrix}$$

Since 1 and 2 work, T is a linear transformation.

#5) (1) $T\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\right) = tr\left(\begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nn}+b_{nn} \end{bmatrix}\right)$

$= (a_{11}+b_{11}) + (a_{22}+b_{22}) + \cdots + (a_{nn}+b_{nn})$

while

equal!!

$T\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) + T\left(\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}\right) = (a_{11}+a_{22}+\cdots+a_{nn}) + (b_{11}+b_{22}+\cdots+b_{nn})$

(2) $T\left(\alpha\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{nn} & & a_{n1} \end{bmatrix}\right) = \alpha a_{11} + \alpha a_{22} + \cdots + \alpha a_{nn}$

while

equal!!

$\alpha T\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}\right) = \alpha(a_{11} + a_{22} + \cdots + a_{nn})$

#6) <u>not</u> <u>linear</u>

Consider $n=2$.

$T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1 \cdot 4$ while $T\left(\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}\right) = 0$

and $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$

So it is not true that

$$T\Big(\underbrace{\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}\Big) = T\left(\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$