Homework 6 MATH 3520 Fall 2019

§6.2

#5 $\{x, 1+x\}$

Test for independence: $\alpha_1 x + \alpha_2(1+x) = \underbrace{0+0x}_{\vec{0} \text{ in } \mathbb{P}_1}$

$(\alpha_1+\alpha_2)x + \alpha_2 = 0 + 0x$

$$\Rightarrow \begin{cases} \alpha_1+\alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \vec{\alpha} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore this set is linearly independent.

#8 $\{2x,\ x-x^2,\ 1+x^3,\ 2-x^2+x^3\}$ in $\mathbb{P}_3$

Test for independence: $\alpha_1[2x] + \alpha_2(x-x^2) + \alpha_3(2-x^2+x^3) = 0x^3+0x^2+0x+0$

$(\alpha_3)x^3 + (-\alpha_2-\alpha_3)x^2 + (2\alpha_1+\alpha_2)x + 2\alpha_3 = 0x^3+0x^2+0x+0$

$$\Rightarrow \begin{cases} \alpha_3 = 0 \\ -\alpha_2-\alpha_3 = 0 \\ 2\alpha_1+\alpha_2 = 0 \\ 2\alpha_3 = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{\alpha} = \vec{0}$$

$\Rightarrow$ independent

#11 Since the Pythagorean identity in trigonometry says

$$\sin^2 x + \cos^2 x = 1,$$

the set $\{1, \sin^2 x, \cos^2 x\}$ is dependent.

#18) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$ ← is it a basis of $\mathbb{R}^{2\times2}$?

First check if $\mathbb{R}^{2\times2} = \text{span}(\mathcal{B})$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times2}$ and consider the equation

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This leads to a system of eqts whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -1 & b \\ 0 & 1 & -1 & c \\ 1 & 0 & 1 & d \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

↑ α_1-col ↑ α_2 col ↑ α_3 col

↑ "0=1" FALSE!!

Thus the equation has no soln.

Therefore, $\mathbb{R}^{2\times2} \neq \text{span}(\mathcal{B})$

Therefore $\mathcal{B}$ is not a basis of $\mathbb{R}^{2\times2}$.

(note; since $\dim \mathbb{R}^{2\times2} = 4$, this set is "too small" to be a basis!!)

#19) Is $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^{2\times 2}$? (3)

Soln: First check if $\mathbb{R}^{2\times 2} = \text{span } \mathcal{B}$.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Consider

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & -1 & 1 & 1 & b \\ 0 & 1 & 1 & 1 & c \\ 1 & 0 & 1 & -1 & d \end{array}\right] \overset{rref}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{2a-b-c}{2} \\ 0 & 1 & 0 & 0 & \frac{c-b}{2} \\ 0 & 0 & 1 & 0 & \frac{-a+b+c+d}{2} \\ 0 & 0 & 0 & 1 & \frac{a-d}{2} \end{array}\right]$$

where the last column is $\vec{\alpha}$.

Therefore $\mathbb{R}^{2\times 2} = \text{span}(\mathcal{B})$

Is $\mathcal{B}$ independent? From previous calculation with $a=b=c=d=0$, we get soln vector $\vec{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, hence $\mathcal{B}$ is independent.

Therefore $\mathcal{B}$ is a basis for $\mathbb{R}^{2\times 2}$.

#24) $V = \mathcal{P}_2$, $\mathcal{B} = \{1, 1+2x+3x^2\}$

Soln: Check if $\mathcal{P}_2 = \text{span}\,\mathcal{B}$

Let $ax^2+bx+c \in \mathcal{P}_2$ be arbitrary.

Consider

$$\alpha_1(1) + \alpha_2(1+2x+3x^2) = ax^2+bx+c$$

$$\downarrow$$

$$3\alpha_2 x^2 + (2\alpha_2)x + (\alpha_1+\alpha_2) = ax^2+bx+c$$

$$\Rightarrow \begin{cases} 3\alpha_2 = a \longrightarrow \alpha_2 = \frac{a}{3} \\ 2\alpha_2 = b \longrightarrow \alpha_2 = b/2 \\ \alpha_1 + \alpha_2 = c \end{cases}$$

not the same ⇓ NO SOLN ⇓ $\mathcal{B}$ does NOT span $\mathcal{P}_2$ ⇓ $\mathcal{B}$ is NOT a basis for $\mathcal{P}_2$

#25) $V = \mathcal{P}_2$, $\mathcal{B} = \{1, 2-x, 3-x^2, x+2x^2\}$

Soln: Check if $\mathcal{P}_2 = \text{span}\,\mathcal{B}$

Let $ax^2+bx+c \in \mathcal{P}_2$ be arbitrary.

Consider $\boxed{ax^2+bx+c = \alpha_1(1) + \alpha_2(2-x) + \alpha_3(3-x^2) + \alpha_4(x+2x^2)}$

$$\Rightarrow ax^2+bx+c = (-\alpha_3+2\alpha_4)x^2 + (-\alpha_2+\alpha_4)x + (\alpha_1+2\alpha_2+3\alpha_3).$$

$$\Rightarrow \begin{cases} -\alpha_3 + 2\alpha_4 = a \\ -\alpha_2 + \alpha_4 = b \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = c \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 0 & 0 & -1 & 2 & a \\ 0 & -1 & 0 & 1 & b \\ 1 & 2 & 3 & 0 & c \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 8 & 3a+2b+c \\ 0 & 1 & 0 & -1 & -b \\ 0 & 0 & 1 & -2 & -a \end{array}\right]$$

Therefore yes, $\mathcal{P}_2 = \text{span}\,\mathcal{B}$.

Is $\mathcal{B}$ independent?

Set $a=b=c=d=0$ above and solve the boxed equation, leading to augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 8 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \Rightarrow \text{soln vector } \vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} -8\alpha_4 \\ \alpha_4 \\ 2\alpha_4 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 2 \\ 1 \end{bmatrix} \alpha_4$$

free (α_4)

$\alpha_1 + 8\alpha_4 = 0$
$\alpha_2 - \alpha_4 = 0$
$\alpha_3 - 2\alpha_4 = 0$

Since nonzero solutions exist, $\mathcal{B}$ is NOT independent.
Therefore $\mathcal{B}$ is NOT a basis.

#27 Find $\mathcal{B}$-coordinate of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ when $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

__Soln__: We must solve

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 \\ \alpha_3 + \alpha_4 & \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1 \\ \alpha_2 + \alpha_3 + \alpha_4 = 2 \\ \alpha_3 + \alpha_4 = 3 \\ \alpha_4 = 4 \end{cases}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \Rightarrow \vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

Therefore,

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

#29 Find $[3x^2 - x + 2]_{\mathcal{B}}$ when $\mathcal{B} = \{1+x, 1-x, x^2\}$

__Soln__: Must solve

$$\alpha_1(1+x) + \alpha_2(1-x) + \alpha_3 x^2 = 3x^2 - x + 2$$

$$\alpha_3 x^2 + (\alpha_1 - \alpha_2)x + (\alpha_1 + \alpha_2) = 3x^2 - x + 2$$

$$\Rightarrow \begin{cases} \alpha_3 = 3 \\ \alpha_1 - \alpha_2 = -1 \\ \alpha_1 + \alpha_2 = 2 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 2 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore,

$$[3x^2 - x + 2]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ 3/2 \\ 3 \end{bmatrix}$$

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#35 $V = \{ p \in \mathcal{P}_2 : p(1) = 0 \}$

Soln: Let $p = ax^2 + bx + c \in \mathcal{P}_2$. The condition $p(1) = 0$ means that $a + b + c = 0$, and so $a = -b - c$ (can solve for any of the vars)

This means that

$$V = \{ (-b-c)x^2 + bx + c : b, c \in \mathbb{R} \}$$

linear combo: $b(-x^2+x) + c(-x^2+1)$ ← basis vects

So a basis for V is $\mathcal{B} = \{ -x^2 + x, -x^2 + 1 \}$

Therefore the dimension of $V = 2$.

#36 $V = \{ p \in \mathcal{P}_2 : xp'(x) = p(x) \}$

Soln: Let $p = ax^2 + bx + c$. Calculate $p'(x) = 2ax + b$. The condition $xp'(x) = p(x)$ says

$$x(2ax + b) = ax^2 + bx + c$$

$$(2a)x^2 + bx = ax^2 + bx + c$$

$$\Downarrow$$

$$\begin{cases} 2a = a \\ b = b \\ c = 0 \end{cases} \longrightarrow \begin{cases} a = 0 \\ b = b \\ c = 0 \end{cases}$$

Therefore, $V = \{ bx : b \in \mathbb{R} \}$ and a basis for V is $\mathcal{B} = \{x\}$.

Therefore $\dim V = 1$.

#39 $V = \{ A \in \mathbb{R}^{2\times 2} : A\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A \}$

Soln: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the condition $A\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$ says

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \overset{\text{matrix multiply}}{\Longrightarrow} \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

equate components $\Longrightarrow$

$$\begin{cases} a = a + c \longrightarrow c = 0 \\ a + b = b + d \longrightarrow a = d \\ c = c \longrightarrow 0 = 0 \\ c + d = d \longrightarrow d = d \end{cases}$$

b is "free"

$a = d$ and $c = 0$

$$\Rightarrow V = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

So a basis for V could be $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

Thus $\dim V = 2$.