# Homework 5 MATH 3520 Linear Algebra

**#24** $V = \mathbb{R}^{3\times 1}$, $W = \left\{ \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} : a \in \mathbb{R} \right\}$

Soln: It is a subspace. If $\begin{bmatrix} a \\ 0 \\ a \end{bmatrix} \in W$ and $\begin{bmatrix} b \\ 0 \\ b \end{bmatrix} \in W$, then

① $\begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ 0 \\ a+b \end{bmatrix} \in W$, since $a+b \in \mathbb{R}$

② if $\alpha \in \mathbb{R}$ is a scaler, then

$$\alpha \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} \alpha a \\ 0 \\ \alpha a \end{bmatrix} \in W, \text{ since } \alpha a \in \mathbb{R}.$$

Also note that $W \subseteq V$.

Therefore, by Theorem 6.2, W is a subspace of V.

**#26** $V = \mathbb{R}^{3\times 1}$, $W = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

Soln: Not a subspace, because $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W$, but

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \notin W$$

So W is not closed under addition of vectors.

Therefore W is not a subspace (it is not a vector space!!)

**#28** $V = \mathbb{R}^{2\times 2}$, $W = \left\{ \begin{bmatrix} a & b \\ b & 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}$

Soln: It is a subspace. If $\begin{bmatrix} a & b \\ b & 2a \end{bmatrix}, \begin{bmatrix} c & d \\ d & 2c \end{bmatrix} \in W$, then

① $\begin{bmatrix} a & b \\ b & 2a \end{bmatrix} + \begin{bmatrix} c & d \\ d & 2c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ b+d & 2(a+c) \end{bmatrix} \in W$, since it has the correct form of a matrix in W.

② if $\alpha \in \mathbb{R}$, then $\alpha \begin{bmatrix} a & b \\ b & 2a \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha b & 2(\alpha a) \end{bmatrix} \in W$

Also $W \subseteq \mathbb{R}^{2\times 2}$.

Therefore by Theorem 6.2, W is a subspace of V.

**#34** $V = \mathcal{P}_2$, $W = \{bx + cx^2 : b, c \in \mathbb{R}\}$

Soln: It is a subspace. Let $bx+cx^2, dx+ex^2 \in \mathcal{P}_2$. Then,

1. $(bx+cx^2) + (dx+ex^2) = (b+d)x + (c+e)x^2 \in W$, and
2. if $\alpha \in \mathbb{R}$, then

$$\alpha(bx+cx^2) = (\alpha b)x + (\alpha c)x^2 \in W$$

Also note that $W \subseteq \mathcal{P}_2$.

Therefore by Theorem 6.2, $W$ is a subspace of $V$.

**#36** $V = \mathcal{P}_2$, $W = \{a + bx + cx^2 : a, b, c \in \mathbb{R},\ abc = 0\}$

Soln: Not a subspace because $0 + x + x^2 \in W$ and $1 + 0x + x^2 \in W$, but

$$(0+x+x^2) + (1+0x+x^2) = 1 + x + 2x^2 \notin W$$, since here, the product of its coefficients is 2, not zero. Therefore $W$ is not closed under $\oplus$ and so it is not a vector space.

**#46** If $U$ and $W$ are subspaces of $V$, then $U \cap W$ is a subspace of $V$.

Proof: First pick $v_1, v_2 \in U \cap W$. Then $v_1, v_2 \in U$ and $v_1, v_2 \in W$. Since $U$ and $W$ are subspaces of $V$, $v_1 + v_2 \in U$ and $v_1 + v_2 \in W$, and consequently, $v_1 + v_2 \in U \cap W$.

Let $\alpha \in \mathbb{R}$. Since $U$ and $W$ are subspaces of $V$, $\alpha v_1 \in U$ and $\alpha v_1 \in W$. Consequently, $\alpha v_1 \in U \cap W$. Also note that $U \cap W \subset V$.

Therefore by Theorem 6.2, $U \cap W$ is a subspace of $V$. ∎

#53 Is $3-5x-x^2 \in \text{Span}\{1-2x,\ x-x^2,\ -2+3x+x^2\}$ ?

Soln: This is asking if there are weights $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that

$$\underbrace{\alpha_1(1-2x) + \alpha_2(x-x^2) + \alpha_3(-2+3x+x^2)} = 3-5x-x^2$$

Algebra $\Rightarrow$ $$= (-\alpha_2+\alpha_3)x^2 + (-2\alpha_1+\alpha_2+3\alpha_3)x + (\alpha_1-2\alpha_3) = 3-5x-x^2$$

This yields system of equations

$$\begin{cases} -\alpha_2+\alpha_3 = -1 \\ -2\alpha_1+\alpha_2+3\alpha_3 = -5 \\ \alpha_1 \quad -2\alpha_3 = 3 \end{cases}$$

Aug matrix $\Rightarrow$ $$\left[\begin{array}{ccc|c} 0 & -1 & 1 & -1 \\ -2 & 1 & 3 & -5 \\ 1 & 0 & -2 & 3 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rightarrow \alpha_1 - 2\alpha_3 = 3 \rightarrow \alpha_1 = 3+2\alpha_3$

$\rightarrow \alpha_2 - \alpha_3 = 1 \rightarrow \alpha_2 = 1+\alpha_3$

Hence the system has solution vector

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 3+2\alpha_3 \\ 1+\alpha_3 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\alpha_3$$

Therefore, a solution exists and so we say YES,

$$3-5x-x^2 \in \text{Span}\{1-2x,\ x-x^2,\ -2+3x+x^2\}.$$

#62 Does $\mathcal{P}_2 = \text{span}\{1+x+2x^2,\ 2+x+2x^2,\ -1+x+2x^2\}$?

Soln: "Trivially" the span $\subset \mathcal{P}_2$. So we must check whether or not $\mathcal{P}_2 \subset \text{span}\{\sim\}$.

Let $c+bx+ax^2 \in \mathcal{P}_2$ be arbitrary. We need to check whether or not there are weights $\alpha_1, \alpha_2, \alpha_3$ so that

$$c+bx+ax^2 = \alpha_1(1+x+2x^2) + \alpha_2(2+x+2x^2) + \alpha_3(-1+x+2x^2)$$

By ALgebra, RHS simplifies and we get

$$c+bx+ax^2 = (\alpha_1+2\alpha_2-\alpha_3) + (\alpha_1+\alpha_2+\alpha_3)x + (2\alpha_1+2\alpha_2+2\alpha_3)x^2$$

$$\Longrightarrow \begin{cases} \alpha_1+2\alpha_2-\alpha_3 = c \\ \alpha_1+\alpha_2+\alpha_3 = b \\ 2\alpha_1+2\alpha_2+2\alpha_3 = a \end{cases}$$

Aug matrix

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & c \\ 1 & 1 & 1 & b \\ 2 & 2 & 2 & a \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

therefore there is NO SOLN in general (unless $a=0$).

Therefore, we conclude that

$$\mathcal{P}_2 \neq \text{span}\{1+x+2x^2,\ 2+x+2x^2,\ -1+x+2x^2\}$$