Homework 3 MATH 3520 Fall 2019

§2.3

#9 Show $\mathbb{R}^{2\times 1} = \text{span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}\right)$

<u>Soln</u>: Let $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^{2\times 1}$ be arbitrary. We must show that $\begin{bmatrix}a\\b\end{bmatrix} \in \text{span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}\right)$, i.e., that there exist scalars α_1, α_2 so that

$$\underbrace{\alpha_1 \begin{bmatrix}1\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\-1\end{bmatrix}}_{= \begin{bmatrix}\alpha_1+\alpha_2\\ \alpha_1-\alpha_2\end{bmatrix}} = \begin{bmatrix}a\\b\end{bmatrix}$$

In other words, we must solve the system $\begin{cases}\alpha_1 + \alpha_2 = a\\ \alpha_1 - \alpha_2 = b\end{cases}$

Write augmented matrix and put into rref

$$\left[\begin{array}{cc|c}1 & 1 & a\\ 1 & -1 & b\end{array}\right] \overset{r_2^* = r_2 - r_1}{\sim} \left[\begin{array}{cc|c}1 & 1 & a\\ 0 & -2 & b-a\end{array}\right] \overset{r_2^* = -\frac{1}{2}r_2}{\sim} \left[\begin{array}{cc|c}1 & 1 & a\\ 0 & 1 & \frac{a-b}{2}\end{array}\right]$$

$$a - \frac{a-b}{2} = \frac{a}{2} + \frac{b}{2}$$

$$\overset{r_1^* = r_1 - r_2}{\sim} \left[\begin{array}{cc|c}1 & 0 & \frac{a+b}{2}\\ 0 & 1 & \frac{a-b}{2}\end{array}\right]$$

Therefore soln vector is $\vec{\alpha} = \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}\frac{a+b}{2}\\ \frac{a-b}{2}\end{bmatrix}$ and we say YES,

$\mathbb{R}^{2\times 1} = \text{span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}\right)$.

#22 is $\left\{\begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\3\end{bmatrix}\right\}$ a linearly indep set of vectors?

Soln : To answer it, we consider the equation

$$\alpha_1\begin{bmatrix}2\\-1\\3\end{bmatrix}+\alpha_2\begin{bmatrix}-1\\2\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$, which represents a system of eqts.

To solve it, write the augmented matrix

$$\left[\begin{array}{cc|c}2&-1&0\\-1&2&0\\3&3&0\end{array}\right]\underset{r_3^*=r_3-\frac{3}{2}r_1}{\overset{r_2^*=r_2+\frac{1}{2}r_1}{\sim}}\left[\begin{array}{cc|c}2&-1&0\\0&3/2&0\\0&9/2&0\end{array}\right]$$

$$3-\frac{3}{2}(-1) = \frac{\cancel{6}}{2}+\frac{3}{2}$$

$$\underset{\substack{r_3^*=\frac{2}{9}r_3\\ r_1^*=\frac{1}{2}r_1}}{\overset{r_2^*=\frac{2}{3}r_2}{\sim}}\left[\begin{array}{cc|c}1&-1/2&0\\0&1&0\\0&1&0\end{array}\right]$$

$$\underset{r_1^*=r_1+\frac{1}{2}r_2}{\overset{r_3^*=r_3-r_2}{\sim}}\left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right]$$

We get soln $\vec{\alpha}=\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$. From this, we conclude that the only linear combo of $\begin{bmatrix}2\\-1\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\2\\3\end{bmatrix}$ that equals $\begin{bmatrix}0\\0\\0\end{bmatrix}$ comes from $\alpha_1=\alpha_2=0$. This means the set $\left\{\begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\3\end{bmatrix}\right\}$ is a linearly independent set of vectors.

#24) Is $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \right\}$ lin dep or lin indep?

Soln: Consider

$$\alpha_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This describes a linear sys of eqts. Write aug. matrix for it:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \underset{r_3^* = r_3 - \frac{1}{2}r_1}{\overset{r_2^* = r_2 - r_1}{\sim}} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & -2 & -6 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 0 \end{array}\right]$$

$2 - \frac{3}{2} = \frac{1}{2}$

$$\underset{r_2^* = -\frac{1}{2}r_2}{\overset{r_3^* = 2r_3}{\sim}} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

$$\overset{r_3^* = r_3 + r_2}{\sim} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{r_1^* = r_1 - 3r_2}{\sim} \left[\begin{array}{ccc|c} 2 & 0 & -8 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{r_1^* = \frac{1}{2}r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\alpha_1 - 4\alpha_3 = 0 \rightarrow \alpha_1 = 4\alpha_3$

$\alpha_2 + 3\alpha_3 = 0 \rightarrow \alpha_2 = -3\alpha_3$

We get soln $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 4\alpha_3 \\ -3\alpha_3 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \alpha_3$

free varible!

So we can obtain nonzero $\alpha_1, \alpha_2, \alpha_3$ by choosing $\alpha_3 \neq 0$.
Therefore the set of vecters is, by definition, <u>linearly dependent</u>.

#30) lin dep or lin indep? $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

<u>Soln</u>: As before, we need to solve a system. Jumping straight to aug matrix, we get

with <u>4 vars</u>

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \underset{r_2 \leftrightarrow r_3}{\overset{r_1 \leftrightarrow r_4}{\sim}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

$$\begin{array}{c} r_4^* = \frac{1}{4} r_4 \\ \sim \\ r_3^* = \frac{1}{3} r_3 \\ r_2^* = \frac{1}{2} r_2 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} r_3^* = r_3 - r_4 \\ \sim \\ r_2^* = r_2 - r_4 \\ r_1^* = r_1 - r_4 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} r_2^* = r_2 - r_3 \\ \sim \\ r_1^* = r_1 - r_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} r_1^* = r_1 - r_2 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Thus soln vector is $\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so the set is <u>linearly independent</u>.