MATH 3520 Fall 2019 HW 13

Problem B: $\{1, x, x^2, x^3, \dots\}$

$$\vec{u}_1 = 1$$

$$\vec{u}_2 = x - \text{proj}_1(x) = x - \frac{\langle 1, x\rangle}{\langle 1,1\rangle} 1$$

$$= x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx} 1$$

$$= x - \frac{x^2/2 \big|_{-1}^{1}}{x \big|_{-1}^{1}} = x - \frac{\overbrace{\cancel{\tfrac{1}{2} - \tfrac{1}{2}}}^{=0}}{2} = x$$

$$\vec{u}_3 = x^2 - \text{proj}_1(x^2) - \text{proj}_x(x^2)$$

$$= x^2 - \frac{\langle 1, x^2\rangle}{\langle 1,1\rangle} 1 - \frac{\langle x, x^2\rangle}{\langle x, x\rangle} x = x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx} - \frac{\overbrace{\int_{-1}^{1} x^3\,dx}^{\to 0}}{\int_{-1}^{1} x^2\,dx} x$$

$$= x^2 - \frac{2/3}{2} - 0$$

$$= x^2 - \frac{1}{3}$$

$$\vec{u}_4 = x^3 - \text{proj}_1(x^3) - \text{proj}_x(x^3) - \text{proj}_{x^2-\frac{1}{3}}(x^3) = x^3 - \cancel{\frac{\overbrace{\langle 1, x^3\rangle}^{0}}{\langle 1,1\rangle}} - \frac{\langle x, x^3\rangle}{\langle x, x\rangle} x - \cancel{\frac{\overbrace{\langle x^2 - \frac{1}{3}, x^3\rangle}^{=0}}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle}} (x^2 - \tfrac{1}{3})$$

$$= x^3 - \frac{2/5}{2/3} x = x^3 - \frac{3}{5} x$$

Problem D

$$P=\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix} \longrightarrow P^{-1}=\frac{1}{1+2}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

$$D=P^{-1}AP=\frac{1}{3}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}\begin{bmatrix}5 & -3\\ -6 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}-1 & -1\\ -16 & 8\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}1 & 0\\ 0 & 24\end{bmatrix}=\begin{bmatrix}1/3 & 0\\ 0 & 8\end{bmatrix}$$

Therefore, $A=PDP^{-1}$ and so $(PDP^{-1})^k=PD^kP^{-1}$ and $D^k=\begin{bmatrix}(1/3)^k & 0\\ 0 & 8^k\end{bmatrix}$

$$\exp(A)=\exp(PDP^{-1})=\sum_{k=0}^{\infty}\frac{(PDP^{-1})^k}{k!}$$

$$=P\left[\sum_{k=0}^{\infty}\frac{D^k}{k!}\right]P^{-1}$$

$$=P\exp(D)P^{-1}$$

$$=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}e^{1/3} & 0\\ 0 & e^{8}\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}e^{1/3} & -e^{8}\\ 2e^{1/3} & e^{8}\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}e^{1/3}+2e^{8} & e^{1/3}-e^{8}\\ 2e^{1/3}-2e^{8} & 2e^{1/3}+e^{8}\end{bmatrix}$$

Similarly,

$$\sin(A)=\sin(PDP^{-1})=\sum_{k=0}^{\infty}\frac{(-1)^k(PDP^{-1})^{2k+1}}{(2k+1)!}=P\left[\sum_{k=0}^{\infty}\frac{(-1)^kD^{2k+1}}{(2k+1)!}\right]P^{-1}$$

$$=P\sin(D)P^{-1}=$$

$$=\frac{1}{3}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}\sin(1/3) & 0\\ 0 & \sin(8)\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}\sin(1/3) & -\sin(8)\\ 2\sin(1/3) & \sin(8)\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}\sin(1/3)+2\sin(8) & \sin(1/3)-\sin(8)\\ 2\sin(1/3)-2\sin(8) & 2\sin(1/3)+\sin(8)\end{bmatrix}$$