HW10 MATH 3520 Fall 2019

§4.1

#6 Show $\vec{v}$ is an e-vector of A + find its e-val.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

e-vector

$$A\vec{x} = \lambda \vec{x}$$

e-value

Soln:

$$A\vec{v} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

hence 0 is the evalue corresponding to evector $\vec{v}$.

#9

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 4 \\ -1 & 5-\lambda \end{bmatrix}\right)$$

$$= -\lambda(5-\lambda) - 4(-1)$$

$$= \lambda^2 - 5\lambda + 4$$

$$= (\lambda - 4)(\lambda - 1)$$

$\Rightarrow$ eigenvalues are $\lambda = 4, 1$

this is what #9 asked for

eigenvector associated to $\lambda = 1$

$$A\vec{x} = \vec{x} \quad \text{— let } \vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 4b \\ -a+5b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{cases} 4b = a & (i) \\ -a+5b = b & (ii) \end{cases} \Rightarrow$$

Plug (i) into (ii):

$-4b + 5b = b \rightarrow b = b$

always true

Therefore, eigenvectors are of form $\vec{x} \in \left\{ \begin{bmatrix} 4b \\ b \end{bmatrix} : b \in \mathbb{R} \right\}$,

taking $b = 1$ yields eigenvector $\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

#11 $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, $\lambda = -1$

Soln: By finding a vector $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $A\vec{x} = -\vec{x}$, we will accomplish our goal.

So that equation becomes

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$$

$$\begin{bmatrix} a+2c \\ -a+b+c \\ 2a+c \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$$

$$\Big\downarrow \text{ add } \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 2a + 2c \\ -a + 2b + c \\ 2a + 2c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \underset{\sim}{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} a+c=0 \rightarrow a=-c \\ b+c=0 \rightarrow b=-c \end{array} \Rightarrow a=b=-c$$

So e-vectors take from $\left\{ \begin{bmatrix} -c \\ -c \\ c \end{bmatrix} : c \in \mathbb{R} \right\}$, hence

taking, say, $c=1$ yields $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, an eigenvector.

§4.2 | #3 | Along the first row...

$$\det \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 1 \det \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} - (-1)\det \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + 0$$

$$= 1(0-1) + 1(1-0)$$

$$= -1 + 1 = 0$$

Along first column...

$$\det \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = 1 \det \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} - (-1)\det \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} + 0$$

$$= 1(0-1) + 1(1-0)$$

$$= 0$$

#10 |

$$\det \begin{bmatrix} \cos(\theta) & \sin(\theta) & \tan(\theta) \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = \cos(\theta) \det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} + 0 + 0$$

$$= \cos(\theta) \underbrace{\left[\cos^2(\theta) - (-\sin^2(\theta))\right]}_{=1}$$

$$= \cos(\theta)$$

#47 | $\det(AB) = \det(A)\det(B) = 3(-2) = -6$

#49 | $\det(B^{-1}A) = \det(B^{-1})\det(A) = \dfrac{\det(A)}{\det(B)} = \dfrac{3}{-2}$

§4.3

#2 $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

a) chr poly: $\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right)$

$$= (-\lambda)(2-\lambda) - (1)(-1)$$
$$= \lambda^2 - 2\lambda + 1$$

b) e-vals: Solve

$$0 = \lambda^2 - 2\lambda + 1$$
$$0 = (\lambda - 1)^2 \longrightarrow \lambda = 1 \quad \text{(double root)}$$
$$\text{(alg mult 2)}$$

c) $\underline{\lambda = 1}$

$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ so that $A\vec{x} = 1\vec{x}$

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 2a+b \\ -a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \begin{cases} 2a+b = a \\ -a = b \end{cases}$$

$$\longrightarrow \begin{cases} a+b=0 \\ a=-b \end{cases}$$

$$\rightarrow \begin{cases} a=-b \\ a=-b \end{cases}$$

Therefore, eigenspace for $\lambda = 1$ is

$$E_1 = \left\{ \begin{bmatrix} -b \\ b \end{bmatrix} : b \in \mathbb{R} \right\} \leftarrow \text{has basis } \mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$\Downarrow$

geom

d) From above,

alg mult $= 2$

geo. mult $= \dim E_1 = 1$

#3) $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

a) $$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)\det\left(\begin{bmatrix} -1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) - (-1)\det\begin{bmatrix} 2 & 0 \\ 1 & 1-\lambda \end{bmatrix} + 0$$

$$= (1-\lambda)\left[(-1-\lambda)(1-\lambda) - 1\right] + \left[2(1-\lambda) - 0\right]$$

$$= (1-\lambda)\left[-1+\cancel{\lambda}\cancel{-\lambda}+\lambda^2-1\right] + 2 - 2\lambda$$

$$= \cancel{-1}+\lambda^2\cancel{-1}+\cancel{\lambda}-\lambda^3+\cancel{\lambda}+\cancel{2}-\cancel{2\lambda}$$

$$= -\lambda^3+\lambda^2$$

b) $0 = \det(A-\lambda I) = -\lambda^3+\lambda^2$

$0 = \lambda^2(-\lambda+1) \rightarrow \lambda = 0, 1$

$\lambda = 0$: alg mult 2 $\quad$ $\lambda = 1$: alg mult 1

c) $\underline{\lambda = 0}$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a+2b \\ -a-b+c \\ b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \overset{\text{rref}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{matrix} a-2c=0 \\ b+c=0 \end{matrix}$$

$$\rightarrow \begin{matrix} a = 2c \\ b = -c \end{matrix}$$

$\Rightarrow$ eigenspace is

$$E_0 = \left\{\begin{bmatrix} 2c \\ -c \\ c \end{bmatrix} : c \in \mathbb{R}\right\} \longleftarrow \text{has basis } \mathcal{B} = \left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$\underline{\lambda = 1}$

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} a + 2b \\ -a - b + c \\ b + c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 2b \\ -a + c \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} b = 0 \\ -a + c = 0 \longrightarrow c = a \end{matrix}$$

Thus eigenspace is

$$E_1 = \left\{ \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} : a \in \mathbb{R} \right\} \longleftarrow \text{has basis } \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

d)

	alg mult	geo mult
$\lambda = 0$:	2	$\dim E_0 = 1$
$\lambda = 1$:	1	$\dim E_1 = 1$