MATH 3520 - EXAM 1 FALL 2019
SOLUTION

Wednesday, 18 September
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (12 points) Solve the system in any way you choose. Don't forget to express the solution in the appropriate way.

   (a) (6 points) $\begin{cases} x_1 & & +2x_3 & = 5 \\ x_1 & +x_2 & +3x_3 & = 6 \\ 2x_1 & & +5x_3 & = 12 \end{cases}$

   Solution: Write an augmented matrix and put it into reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 1 & 1 & 3 & 6 \\ 2 & 0 & 5 & 12 \end{bmatrix} \overset{r_2^*=r_2-r_1, r_3^*=r_3-2r_1}{\sim} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\overset{r_2^*=r_2-r_3, r_1^*=r_1-2r_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   Therefore the solution of the system is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

   (b) (6 points) $\begin{cases} x_1 & & +x_3 & = 2 \\ -2x_1 & +x_2 & -x_3 & = -5 \\ -x_1 & +x_2 & & = -3 \end{cases}$

   Solution: Write an augmented matrix and put it into reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ -2 & 1 & -1 & -5 \\ -1 & 1 & 0 & -3 \end{bmatrix} \overset{r_2^*=r_2+2r_1, r_3^*=r_3+r_1}{\sim} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$
$$\overset{r_3^*=r_3-r_2}{\sim} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   This matrix encodes the equations

$$\begin{cases} x_1 + x_3 = 2 & \longrightarrow x_1 = 2 - x_3 \\ x_2 + x_3 = -1 & \longrightarrow x_2 = -1 - x_3 \end{cases}$$

   Therefore we have solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ -1 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3,$$

   where x_3 is a free variable.

2. (16 points) If $\vec{v}$ a linear combination of $\vec{u}_1$ and $\vec{u}_2$? If so, explain why. If not, explain why not.

   (a) (8 points) $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$

   Solution: We are asked if the following equation has a solution:

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Algebra on the left-hand side yields the equivalent equation

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 \\ 4\alpha_2 \\ 3\alpha_1 - 2\alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Therefore we solve it by putting an augmented matrix into reduced echelon form:

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 2 \\ 3 & -2 & 2 \end{bmatrix} \overset{r_3^*=r_3-3r_1}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 2 \\ 0 & -8 & -4 \end{bmatrix}$$
$$\overset{r_3^*=r_3+2r_2}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\overset{r_2^*=\frac{1}{4}r_2}{\sim} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$
$$\overset{r_1^*=r_1-2r_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the equation has solution

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix},$$

and so we say **YES**, $\vec{v}$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

(b) (8 points) $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Solution: We consider the equation

$$\alpha_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This gives augmented matrix

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \overset{r_2^*=r_2+r_1}{\sim} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$
$$\overset{r_3^*=r_3+r_2}{\sim} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$

but the third row of this matrix encodes the equation "$0 = 3$" which is false. Therefore we say **NO**, $\vec{v}$ is not a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

3. (12 points) Demonstrate that $\mathbb{R}^{2\times 1} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$.

Solution: It is "obvious" that $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\} \subset \mathbb{R}^{2\times 1}$. Therefore we must show that

$$\mathbb{R}^{2\times 1} \subset \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}.$$

Let $A = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{2\times 1}$ be arbitrary. Consider the equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Solve it by using the agumented matrix

$$\begin{bmatrix} 1 & -1 & a \\ 2 & 0 & b \end{bmatrix} \overset{r_2^*=r_2-2r_1}{\sim} \begin{bmatrix} 1 & -1 & a \\ 0 & 2 & b-2a \end{bmatrix} \overset{r_2^*=\frac{1}{2}r_2}{\sim} \begin{bmatrix} 1 & -1 & a \\ 0 & 1 & \frac{b}{2}-a \end{bmatrix} \overset{r_1^*=r_1+r_2}{\sim} \begin{bmatrix} 1 & 0 & \frac{b}{2}+\frac{a}{2} \\ 0 & 1 & \frac{b}{2}-a \end{bmatrix},$$

therefore the weights

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{b+a}{2} \\ \frac{b}{2}-a \end{bmatrix}$$

demonstrate that we may arrive at the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and the desired subset relation (and hence equality) is now established.

4. (18 points) Is the set of vectors linearly independent or linearly dependent? Explain why.

(a) (9 points) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution: Consider the equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 5 \\ 9 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve it, we write the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ 9 & 1 & 1 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From this we see that the solution is

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and so we conclude that the set in question is **linearly independent**.

(b) (9 points) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix} \right\}$

Solution: Consider the equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ -3 \\ -7 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve it, write the augmented matrix

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ 2 & -3 & 7 & 0 \\ 3 & -7 & 8 & 0 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

From this we see that the solution is

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and so we conclude that the set in question is **linearly independent**.

5. (18 points) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} -1 & 0 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Perform the computation or state why it is impossible to compute.

   (a) (6 points) $2A + B$
   Solution: Compute

$$\begin{aligned} 2A + B &= 2\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 2 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2-1 & 4+0 \\ 2+2 & -2+5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 4 & 3 \end{bmatrix}. \end{aligned}$$

   (b) (6 points) $B + D$
   Solution: This is impossible to compute because $B \in \mathbb{R}^{2\times 2}$ while $D \in \mathbb{R}^{2\times 4}$ and so the addition is not defined since they are in difference spaces.

   (c) (6 points) $\left(AC^T\right)C$
   Solution: Compute

$$\begin{aligned} \left(AC^T\right)C &= \left(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}^T\right) \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}\right) \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1\cdot 1 + 2\cdot(-1) & 1\cdot 0 + 2\cdot(-1) & 1\cdot(-1) + 2\cdot 1 \\ 1\cdot 1 + (-1)\cdot(-1) & 1\cdot 0 + (-1)\cdot(-1) & 1\cdot(-1) + (-1)\cdot 1 \end{bmatrix}\right) \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (-1)\cdot 1 + (-2)\cdot 0 + 1\cdot(-1) & (-1)\cdot(-1) + (-2)\cdot(-1) + 1\cdot 1 \\ 2\cdot 1 + 1\cdot 0 + (-2)\cdot(-1) & 2\cdot(-1) + 1\cdot(-1) + (-2)\cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ 4 & -5 \end{bmatrix}. \end{aligned}$$

6. (12 points) A matrix A is called idempotent if the following equation holds:

$$A^2 = A,$$

where A^2 means AA – i.e. multiplying A with itself. Are the following matrices idempotent? Explain why or why not.

(a) (6 points) $\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$

Solution: Compute

$$\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 3 + (-6) \cdot 1 & 3 \cdot (-6) + (-6) \cdot (-2) \\ 1 \cdot 3 + (-2) \cdot 1 & 1 \cdot (-6) + (-2) \cdot (-2) \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix},$$

therefore this matrix **IS** idempotent.

(b) (6 points) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Solution: Compute

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

therefore this matrix **IS NOT** idempotent.

7. (12 points) Find the inverse of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$.

Solution: Set up the appropriate matrix and put it into reduced echelon form:

$$\begin{bmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$