HW8 MATH 3504 Spring 2019

p-190 #3a) $\begin{cases} x' = x & (i) \\ y' = x + 2y & (ii) \end{cases}$

Soln: Notice (i) does not depend on y, so we solve it directly (by separation of vars):

$$\frac{dx}{dt} = x \longrightarrow \int \frac{1}{x}\,dx = \int 1\,dt$$

$$\longrightarrow \ln(x) = t + C$$

$$\longrightarrow x = c_1 e^t$$

Now (ii) becomes

$$y' = c_1 e^t + 2y$$

$$y' - 2y = c_1 e^t \quad \longleftarrow \text{linear 1st order } y' + py = f$$

integrating factor: $\mu = e^{\int -2} = e^{-2t}$ $\Rightarrow$ int factor $\mu = e^{\int p}$

Mult by μ to get

$$(ye^{-2t})' = c_1 e^{-t}$$

$$\downarrow \int$$

$$ye^{-2t} = -c_1 e^{-t} + c_2$$

$$\downarrow \text{mult by } e^{2t}$$

$$y = -c_1 e^t + c_2 e^{-2t}$$

Therefore soln is

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ -c_1 e^t + c_2 e^{-2t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}$$

#3c $\begin{cases} x' = x + 2y & \text{(i)} \\ y' = x & \text{(ii)} \end{cases}$

Soln: Take $\frac{d}{dt}$ of (i) to get

$$x'' = x' + 2y'$$

Use (i) to replace y' with x to get

$$x'' = x' + 2x \longrightarrow x'' - x' - 2x = 0$$

➡ guess $x = e^{\lambda t}$ + plug in to get

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\boxed{\lambda = 2, -1}$$

$$\Longrightarrow x(t) = c_1 e^{2t} + c_2 e^{-t}$$

Since $y' = x = c_1 e^{2t} + c_2 e^{-t}$, then integration shows

$$y = \int c_1 e^{2t} + c_2 e^{-t}\, dt$$

$$= \frac{c_1}{2} e^{2t} - c_2 e^{-t}$$

Therefore, solution vector is

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-t} \\ \frac{c_1}{2} e^{2t} - c_2 e^{-t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} e^{2t} \\ \frac{1}{2} e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

#3d $\begin{cases} x' = -x - 2y & (i) \\ y' = 2x - y & (ii) \end{cases}$

Soln: Take $\frac{d}{dt}$ of (i) to get

$$x'' = -x' - 2y'$$

Use (ii) to get

$$x'' = -x' - 2(2x - y)$$

Solve (i) for y to get $y = -\frac{x}{2} - \frac{x'}{2}$ + plug in to arrive at

$$x'' = -x' - 2\left(2x - \left(-\frac{x}{2} - \frac{x'}{2}\right)\right)$$

$$x'' = -x' - 4x + \left(x + x'\right)$$

$$x'' + 2x' + 5x = 0$$

$$\downarrow x = e^{\lambda t}$$

$$\lambda^2 + 2\lambda + 5 = 0 \longrightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

$$= -1 \pm \frac{1}{2}\sqrt{-16}$$

$$= -1 \pm \frac{1}{2}(4i)$$

$$= -1 \pm 2i$$

Therefore,

$$x = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

and by (ii), we saw $y = -\frac{x}{2} - \frac{x'}{2}$, so compute

$$x' = c_1\left[-e^{-t}\cos(2t) + e^{-t}(-\sin(2t))(2)\right] + c_2\left[-e^{-t}\sin(2t) + 2e^{-t}\cos(2t)\right],$$

so

$$y = -\left(\frac{c_1}{2}e^{-t}\cos(2t) + \frac{c_2}{2}e^{-t}\sin(2t)\right) - \left(\begin{array}{l} \frac{c_1}{2}\left[-e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right] \\ + \frac{c_2}{2}\left[-e^{-t}\sin(2t) + 2e^{-t}\cos(2t)\right] \end{array}\right)$$

$$= c_1\left[\frac{-e^{-t}}{2}\cos(2t) - \frac{1}{2}\left[-e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right]\right]$$

$$+ c_2\left[\frac{-e^{-t}\sin(2t)}{2} - \frac{1}{2}\left[-e^{-t}\sin(2t) + 2e^{-t}\cos(2t)\right]\right]$$

$$= \frac{e^{-t}}{2}\left\{c_1\left[-\cos(2t) + \cos(2t) + 2\sin(2t)\right]\right.$$

$$\left. + c_2\left[-\sin(2t) + \sin(2t) - 2\cos(2t)\right]\right\}$$

$$= c_1 e^{-t}\sin(2t) - c_2 e^{-t}\cos(2t)$$

Therefore we have solution vector

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) \\ c_1 e^{-t}\sin(2t) \quad -c_2 e^{-t}\cos(2t) \end{pmatrix}$$

$$= c_1 e^{-t}\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix}$$

p. 198 #1 $A=\begin{pmatrix}1&3\\2&4\end{pmatrix}$, $B=\begin{pmatrix}-1&0\\3&7\end{pmatrix}$, $\vec{x}=\begin{pmatrix}2\\-5\end{pmatrix}$

$$A+B=\begin{pmatrix}1&3\\2&4\end{pmatrix}+\begin{pmatrix}-1&0\\3&7\end{pmatrix}=\begin{pmatrix}1+(-1)&3+0\\2+3&4+7\end{pmatrix}=\begin{pmatrix}0&3\\5&11\end{pmatrix}$$

$$B-4A=\begin{pmatrix}-1&0\\3&7\end{pmatrix}-4\begin{pmatrix}1&3\\2&4\end{pmatrix}=\begin{pmatrix}-1&0\\3&7\end{pmatrix}+\begin{pmatrix}-4&-12\\-8&-16\end{pmatrix}$$

$$=\begin{pmatrix}-1-4&0-12\\3-8&7-16\end{pmatrix}$$

$$=\begin{pmatrix}-5&-12\\-5&-9\end{pmatrix}$$

$$AB=\begin{pmatrix}1&3\\2&4\end{pmatrix}\begin{pmatrix}-1&0\\3&7\end{pmatrix}=\begin{pmatrix}(1)(-1)+3(3)&(1)(0)+3(7)\\2(-1)+4(3)&2(0)+4(7)\end{pmatrix}$$

$$=\begin{pmatrix}-1+9&0+21\\-2+12&0+28\end{pmatrix}$$

$$=\begin{pmatrix}8&21\\10&28\end{pmatrix}$$

$$BA=\begin{pmatrix}-1&0\\3&7\end{pmatrix}\begin{pmatrix}1&3\\2&4\end{pmatrix}=\begin{pmatrix}(-1)(1)+0(2)&(-1)(3)+0(4)\\3(1)+7(2)&3(3)+7(4)\end{pmatrix}$$

$$=\begin{pmatrix}-1+0&-3+0\\3+14&9+28\end{pmatrix}$$

$$=\begin{pmatrix}-1&-3\\17&37\end{pmatrix}$$

$$A^2=A\cdot A=\begin{pmatrix}1&3\\2&4\end{pmatrix}\begin{pmatrix}1&3\\2&4\end{pmatrix}=\begin{pmatrix}(1)(1)+3(2)&1(3)+3(4)\\2(1)+4(2)&2(3)+4(4)\end{pmatrix}$$

$$=\begin{pmatrix}1+6&3+12\\2+8&6+16\end{pmatrix}$$

$$=\begin{pmatrix}7&15\\10&22\end{pmatrix}$$

$$B\vec{x}=\begin{pmatrix}-1&0\\3&7\end{pmatrix}\begin{pmatrix}2\\-5\end{pmatrix}=\begin{pmatrix}(-1)(2)+0(-5)\\3(2)+7(-5)\end{pmatrix}=\begin{pmatrix}-2+0\\6-35\end{pmatrix}=\begin{pmatrix}-2\\-29\end{pmatrix}$$

$$AB\vec{x}=(AB)\vec{x}=\begin{pmatrix}8&21\\10&28\end{pmatrix}\begin{pmatrix}2\\-5\end{pmatrix}=\begin{pmatrix}8(2)+21(-5)\\10(2)+28(-5)\end{pmatrix}=\begin{pmatrix}16-105\\20-140\end{pmatrix}$$

$$=\begin{pmatrix}-89\\-120\end{pmatrix}$$

$\det \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3/2 \\ 1 & -1/2 \end{pmatrix}$$

$$\det B = \det \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} = (-1)(7) - (0)(3) = -7 - 0 = -7$$

$$B^3 = (B^2)(B) = \left[\begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix} \right] \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 1+0 & 0+0 \\ -3+21 & 0+49 \end{pmatrix} \right] \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 18 & 49 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} -1+0 & 0+0 \\ -18+147 & 0+343 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 129 & 343 \end{pmatrix}$$

$$AI = A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} 7 & 0 \\ -3 & -1 \end{pmatrix} = \frac{1}{-7} \begin{pmatrix} 7 & 0 \\ -3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 3/7 & 1/7 \end{pmatrix}$$