HW5 MATH 3504 Spring 2019

p.120 #1c) Solve $t^2x'' + 3tx' + x = 0$

Soln: Cauchy-Euler $\rightarrow$ guess $x = t^m$

$$x' = mt^{m-1}$$

$$x'' = m(m-1)t^{m-2}$$

Plug in guess to get

$$t^2(m(m-1)t^{m-2}) + 3t(mt^{m-1}) + t^m = 0$$

$$m(m-1) + 3m + 1 = 0$$

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1 \ \boxed{\text{double root}}$$

$$\Rightarrow \ x(t) = c_1 \frac{1}{t} + c_2 \frac{1}{t} \ln(t)$$

1f) $\begin{cases} t^2x'' + 3tx' - 8x = 0 \\ x(1) = 0, \ x'(1) = 2 \end{cases}$

Soln: Guess $x = t^m$ & plug in to get

$$m(m-1) + 3m - 8 = 0$$

$$m^2 + 2m - 8 = 0$$

$$(m+4)(m-2) = 0$$

$$m = 2, -4$$

$$\Rightarrow \begin{cases} x(t) = c_1 t^2 + \dfrac{c_2}{t^4} \\ x'(t) = 2c_1 t - \dfrac{4c_2}{t^5} \end{cases}$$

Plug in initial conditions:

$$\begin{cases} 0 = x(1) = c_1 + c_2 \rightarrow c_1 = -c_2 \\ 2 = x'(1) = 2c_1 - 4c_2 \end{cases}$$

$$2 = -2c_2 - 4c_2$$

$$2 = -6c_2$$

$$\boxed{c_2 = -\frac{1}{3}} \longrightarrow \boxed{c_1 = \frac{1}{3}}$$

Therefore we get

$$x(t) = \frac{1}{3}t^2 - \frac{1}{3}\cdot\frac{1}{t^4}$$

p.124 #1a| $x'' + x = \tan(t)$

Soln: First solve homogeneous: $x'' + x = 0$

$$\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$$

$$\underline{\text{homogen soln}} \Rightarrow x(t) = c_1 \underbrace{\cos(t)}_{x_1(t)} + c_2 \underbrace{\sin(t)}_{x_2(t)}$$

Now we need to find $x_p(t)$. Use variation of parameters (eqt (2.30) on p.121):

Wronskian: $W(t) = \cos^2(t) + \sin^2(t) = 1$

So,

$$x_p(t) \overset{(2.30)}{=} -x_1(t)\int \frac{x_2(t)f(t)}{W(t)}dt + x_2(t)\int \frac{x_1(t)f(t)}{W(t)}dt$$

$$= -\cos(t)\int \sin(t)\tan(t)\,dt + \sin(t)\int \cos(t)\tan(t)\,dt$$

$$= -\cos(t)\int \frac{\sin^2(t)}{\cos(t)}dt + \sin(t)\int \cos(t)\frac{\sin(t)}{\cos t}dt$$

$\sin^2(t)=1-\cos^2(t)$

$$= -\cos(t)\int \frac{1-\cos^2(t)}{\cos(t)}\,dt + \sin(t)\int \sin t\,dt$$

$$= -\cos(t)\int \sec(t)\,dt - \sin(t)\cos(t)$$

(look up)

$$= -\cos(t)\ln\left(\tan(t)+\sec(t)\right) - \sin(t)\cos(t)$$

Therefore the soln is

$$\boxed{x(t) = c_1\cos(t) + c_2\sin(t) - \cos(t)\ln(\tan(t)+\sec(t)) - \sin(t)\cos(t)}$$

$\boxed{\text{p. 124 \#1e}}$ $x'' + x = \frac{1}{t+1}$

homogen soln: $x(t) = c_1\cos(t) + c_2\sin(t)$

var of param: $W(t) = 1$

$$x_p(t) = -\cos(t)\int^t \frac{\sin(\tau)}{1+\tau}\,d\tau + \sin(t)\int^t \frac{\cos(t)}{1+t}\,dt$$

these do not simplify

they are the "sine integral" and "cosine integral" functions

Therefore soln is

$$x(t) = c_1\cos(t) + c_2\sin(t) - \cos(t)\int^t \frac{\sin(\tau)}{1+\tau}\,d\tau + \sin(t)\int^t \frac{\cos(\tau)}{1+\tau}\,d\tau$$