$$c_1 e^{at}\left(\vec{w}\cos(bt) - \vec{z}\sin(bt)\right) + c_2 e^{at}\left(\vec{w}\sin(bt) + \vec{z}\cos(bt)\right)$$

HW11 MATH 3504 Spring 2019

$a=0, b=\pm 1$

p.221 | #1a | eigenpairs $\lambda = \pm i$, $\vec{v} = \begin{pmatrix}1\\ \pm i\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \pm \begin{pmatrix}0\\i\end{pmatrix} = \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\vec{w}} \pm \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\vec{z}} i$

use ⊕ soln

↳ general soln: $\vec{x} = c_1\left(\begin{pmatrix}1\\0\end{pmatrix}\cos(t) - \begin{pmatrix}0\\1\end{pmatrix}\sin(t)\right) + c_2\left(\begin{pmatrix}1\\0\end{pmatrix}\sin(t) + \begin{pmatrix}0\\1\end{pmatrix}\cos(t)\right)$

$$= c_1\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + c_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

#1b | eigenpairs: $\lambda = \underbrace{1 \pm 2i}_{a=1, b=2}$, $\vec{v} = \begin{pmatrix}1\\ 1\pm i\end{pmatrix} = \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\vec{w}} \pm \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\vec{z}} i$

use ⊕ soln

↳ general soln: $\vec{x} = c_1 e^t\left(\begin{pmatrix}1\\1\end{pmatrix}\cos(2t) - \begin{pmatrix}0\\1\end{pmatrix}\sin(2t)\right) + c_2 e^t\left(\begin{pmatrix}1\\1\end{pmatrix}\sin(2t) + \begin{pmatrix}0\\1\end{pmatrix}\cos(2t)\right)$

$$= c_1 e^t\begin{pmatrix}\cos(2t)\\ \cos(2t) - \sin(2t)\end{pmatrix} + c_2 e^t\begin{pmatrix}\sin(2t)\\ \sin(2t) + \cos(2t)\end{pmatrix}$$

#2 | $A = \begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix} \Rightarrow 0 = \det(A - \lambda I) = \det\begin{pmatrix}1-\lambda & -1\\ 1 & 2-\lambda\end{pmatrix}$

$$= (1-\lambda)(2-\lambda) - (1)(-1)$$
$$= 2 - \lambda - 2\lambda + \lambda^2 + 1$$
$$= \lambda^2 - 3\lambda + 3$$

$\vec{x}' = \begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}\vec{x}$

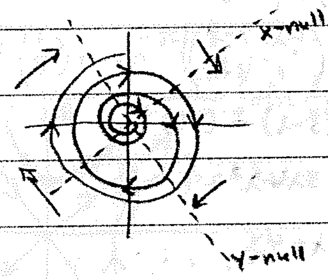
$\begin{cases} x' = x - y \\ y' = x + 2y \end{cases}$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{9 - 4(3)}}{2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-3} = \frac{3}{2} \pm \frac{1}{2}\sqrt{3}\, i$$

$\frac{3}{2} > 0$ ⇓ outwards spiral

x-nullcline: $0 = x - y$, $y = x$

y-nullcline: $0 = x + 2y$, $y = -\frac{1}{2}x$



p.225/ #1a] $\vec{x}\,' = \begin{pmatrix}-3 & 1\\ 0 & -3\end{pmatrix}\vec{x}$

evals $\rightarrow$ $0 = \det\begin{pmatrix}-3-\lambda & 1\\ 0 & -3-\lambda\end{pmatrix} = (-3-\lambda)(-3-\lambda)$

$\Rightarrow \lambda = -3$ double root

Find evectors:

$$\left(\begin{pmatrix}-3 & 1\\ 0 & -3\end{pmatrix} - \begin{pmatrix}-3 & 0\\ 0 & -3\end{pmatrix}\right)\vec{v} = \vec{0}$$

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \rightarrow v_2 = 0$$

$\Rightarrow \vec{v} = \begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}v_1 \Rightarrow \vec{v} = \begin{pmatrix}1\\ 0\end{pmatrix}$ is an evector

To find 2nd, $\vec{w}$, solve

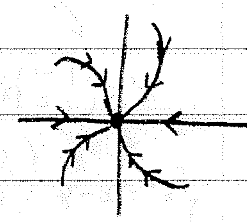
$$(A-\lambda I)\vec{w} = \vec{v} \Rightarrow \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$\Rightarrow w_2 = 1$ (w_1 free) $w_1 = 0$

$$\Rightarrow \vec{w} = \begin{pmatrix}w_1\\ w_2\end{pmatrix} = \begin{pmatrix}w_1\\ 1\end{pmatrix} \rightsquigarrow \vec{w} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

Therefore, general soln can be written as (for $x_2 = e^{\lambda t}(t\vec{v}+\vec{w})$)

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$
$$= c_1\begin{pmatrix}1\\ 0\end{pmatrix}e^{-3t} + c_2\left(t\begin{pmatrix}1\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\right)e^{-3t}$$
$$= c_1\begin{pmatrix}1\\ 0\end{pmatrix}e^{-3t} + c_2\begin{pmatrix}t\\ 1\end{pmatrix}e^{-3t}$$



#1b] $\vec{x}\,' = \begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}\vec{x}$

evals $\rightarrow$ $0 = \det\begin{pmatrix}1-\lambda & -1\\ 1 & 3-\lambda\end{pmatrix}$

$= (1-\lambda)(3-\lambda) - (-1)$

$= 3 - \lambda - 3\lambda + \lambda^2 + 1$

$= \lambda^2 - 4\lambda + 4$

$= (\lambda-2)^2$

$\Rightarrow \lambda = 2$ double root

eigenvectors $\left(\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix} - \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\right)\vec{v} = \vec{0}$

$$\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = 0 \rightarrow -v_1 - v_2 = 0 \rightarrow v_2 = -v_1$$

free, take = 1

$\Rightarrow \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} v_1$

$\Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For 2nd soln, $\vec{w}$, solve $(A - \lambda I)\vec{w} = \vec{v}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\rightarrow -w_1 - w_2 = 1$

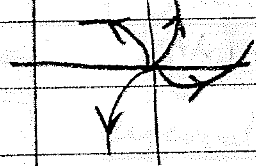
$\rightarrow w_2 = -w_1 - 1$ w_1 free, take = 0

$\rightarrow \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ -w_1 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\Rightarrow$ gen soln is

$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) e^{2t}$

$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} t \\ -t-1 \end{pmatrix} e^{2t}$



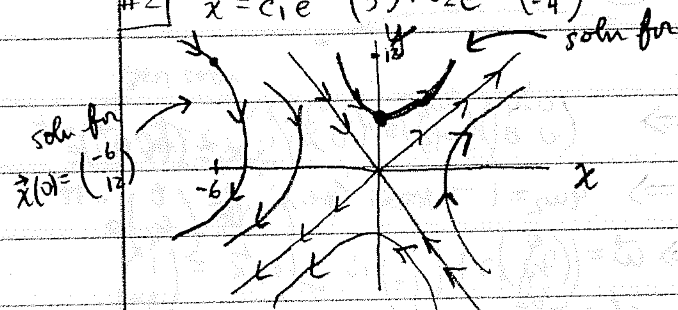
eigenpairs

p.237 #1 $A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \rightsquigarrow \lambda = 3, \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda = 1, \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \rightsquigarrow \lambda = 8, \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\lambda = 0, \vec{v} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$C = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix} \rightsquigarrow \lambda = 2i, \vec{v} = \begin{pmatrix} 2+2i \\ 1 \end{pmatrix}$ and $\lambda = 2-2i, \vec{v} = \begin{pmatrix} 2-2i \\ 1 \end{pmatrix}$

#2 $\vec{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ $\vec{x}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

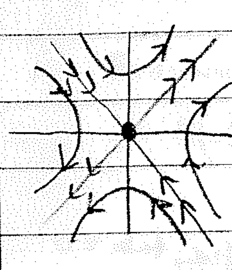


eigenpairs

\#4a) $\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x} \rightsquigarrow \lambda = 4, \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $\lambda = -1, \vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Rightarrow$ gen soln is

$$\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$



saddle pt at (0,0)

\#4b) $\vec{x}' = \begin{pmatrix} -3 & 4 \\ 0 & -3 \end{pmatrix} \vec{x}$

eigenvalues : $0 = \det \begin{pmatrix} -3-\lambda & 4 \\ 0 & -3-\lambda \end{pmatrix}$

$= (-3-\lambda)(-3-\lambda) - 0$

$= 9 + 3\lambda + 3\lambda + \lambda^2 = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$

$\Rightarrow \boxed{\lambda = -3 \text{ double root}}$

eigenvectors

$\lambda = -3$

$\left(\begin{pmatrix} -3 & 4 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \right) \vec{v} = \vec{0}$

$\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow 4v_2 = 0 \rightarrow v_2 = 0$

$\Rightarrow \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \underset{v_1 = 1}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Big\} \; x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t}$

2nd eigenvector $\vec{w}$

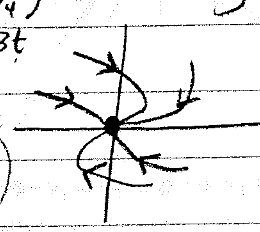
$(A - \lambda I)\vec{w} = \vec{v} \Rightarrow \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow 4w_2 = 1 \rightarrow w_2 = 1/4$

$\Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}$

$\Big\} \Rightarrow x_2 = (t\vec{v} + \vec{w}) e^{-3t} = \begin{pmatrix} t \\ 1/4 \end{pmatrix} e^{-3t}$

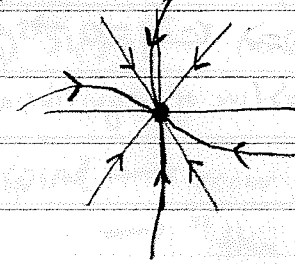
$\Rightarrow \vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} t \\ 1/4 \end{pmatrix} e^{-3t}$



stable node at (0,0)

#4d $\vec{x}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix} \vec{x}$ $\rightsquigarrow$ eigenpairs: $\lambda = -11,\ \vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$; $\lambda = -4,\ \vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\Rightarrow$ gen soln: $\vec{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-11t} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-4t}$



stable node at $(0,0)$

#5 $\begin{cases} \vec{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \vec{x} \\ \vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$

eigenvalues

$$0 = \det \begin{pmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$$
$$= -2\lambda + \lambda^2 - (-1)$$
$$= \lambda^2 - 2\lambda + 1$$
$$= (\lambda - 1)^2$$
$$\Rightarrow \boxed{\lambda = 1 \text{ double root}}$$

eigenvectors

$$\left(\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \vec{v} = \vec{0}$$
$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$v_1 + v_2 = 0 \to v_2 = -v_1$$
$$\Rightarrow \vec{v} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} v_1 \quad (v_1 \text{ free})$$
$$\overset{v_1 = 1}{\Rightarrow} \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2^{nd} eigenvector ($\vec{w}$)

Solve $(A - \lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$w_1 + w_2 = 1$$
$$w_2 = 1 - w_1$$
$$\Rightarrow \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ 1 - w_1 \end{pmatrix}$$
$$\downarrow w_1 = 0$$
$$\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gen soln

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} t \\ 1-t \end{pmatrix} e^{t}$$

$\Downarrow$ initial cond

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 + c_2 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} c_1 = 1 \\ c_2 - c_1 = 1 \end{matrix}}$$

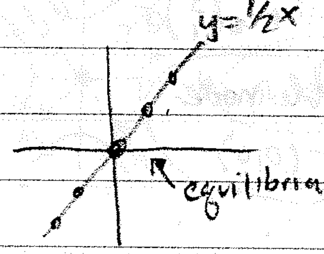
$c_1 = 1$

$c_2 - 1 = 1 \to \boxed{c_2 = 2}$

soln is: $\vec{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + 2(1-t)e$

#6 $\vec{x}' = \begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\vec{x}$ det $= 0$ → equilibrium is a line

a) equilibrium: set $\vec{x}' = 0 \rightarrow \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$



$\Rightarrow x - 2y = 0 \rightarrow y = \frac{1}{2}x$

⇒ line of equilibria

c+b) eigenpairs → $\lambda = 0, \vec{v} = \begin{pmatrix}2\\1\end{pmatrix}$; $\lambda = 5, \vec{v} = \begin{pmatrix}-1\\2\end{pmatrix}$

general soln: $\vec{x} = \underbrace{c_1\begin{pmatrix}2\\1\end{pmatrix}}_{\text{equilibria}} + \underbrace{c_2\begin{pmatrix}-1\\2\end{pmatrix}e^{5t}}_{\text{linear orbits}}$