Intro to nonstandard analysis

HISTORY → Newton/Leib → Berkeley → Cauchy et al → Robinson

Props a "pos infitsml" ϵ should have

- less than any $\oplus$ #
- greater than 0
- $\frac{1}{\epsilon}$ should be an "inf. #"

Let $\mathbb{R}^{\mathbb{N}}$ = set of sequences of reals, i.e. if $x \in \mathbb{R}^{\mathbb{N}}$,

then $x = (x_1, x_2, x_3, \dots)$

Represent real # $w \in \mathbb{R}$ in $\mathbb{R}^{\mathbb{N}}$ by $(w, w, w, w, w, \dots)$

Arithmetic in $\mathbb{R}^{\mathbb{N}}$:

ie, $x = \_\_\_$, $y = \_\_\_$, $z = \_\_\_$, then $z \cdot (x+y) = (z_1(x_1+y_1), \dots)$

i.e. $\exists r^{-1}$ s.t. $rr^{-1} = (1,1,\dots)$

Look at $r \in \mathbb{R}^{\mathbb{N}}$ so $r = (0,1,1,1,\dots)$

This is nonzero, so we would like it to be divisible to get $(1,1,\dots)$

$\Rightarrow$ Problem is 0 is in there...

Q: How can we devise a way to treat $(0,1,1,\dots)$ like $(1,1,\dots)$

A: We say that "almost all" components are $=1$.

define this

Def: Let I be a set. A <u>filter $\mathcal{F}$ over I</u> is a subset of $\mathcal{P}(I)$

s.t...

FAx1 $\emptyset \notin \mathcal{F}$,

FAx2 $\forall A \in \mathcal{F}, \forall B \in \mathcal{F} \ (A \cap B \in \mathcal{F})$

FAx3 $\forall A \in \mathcal{F} \ \forall C \ (A \subseteq C \subseteq I \to C \in \mathcal{F})$

Ex: let $I = \{0,1,2\}$

$$\mathcal{P}(I) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{0,1,2\}, \{1,2\}\}$$

Is $\mathcal{F} = \{\{0\}, \{0,1,2\}\}$ a filter?

No — why not? Fails Ax 3

Is $\mathcal{F} = \{\{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$ a filter?

~~No~~ ~~Fails Ax 3~~

Yes

~~Is $\mathcal{F} = \{\emptyset, \{0\}, \{0,1\}, \{0, \ldots\}, \ldots\}$~~

Ex: If $I = \mathbb{N}$, then $\mathcal{F}^{co} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$ is a filter.

Why? First, $\emptyset \notin \mathcal{F}^{co}$ b/c $\mathbb{N} \setminus \emptyset = \mathbb{N}$ is infinite. (Ax 1)
If $A, B \in \mathcal{F}^{co}$, then $\mathbb{N} \setminus A$ and $\mathbb{N} \setminus B$ are finite.
Therefore, $A \cap B$ has ∞-many elements, missing only those in $(\mathbb{N} \setminus A) \cup (\mathbb{N} \setminus B)$, so $\mathbb{N} \setminus (A \cap B)$ is finite. (Ax 2)
Also if $A \in \mathcal{F}^{co}$ and $A \subseteq C \subseteq \mathbb{N}$, then C is bigger than A and so $\mathbb{N} \setminus C$ is smaller than finite set $\mathbb{N} \setminus A$. Thus $\mathbb{N} \setminus C$ is finite, so $C \in \mathcal{F}^{co}$.

Def: A filter $\mathcal{U}$ with additional axiom

F Ax 4: $\forall A \subseteq I \; (A \in \mathcal{U} \vee I \setminus A \in \mathcal{U})$

is called an ultrafilter.

Ex: $\mathcal{F}^{co}$ is not an ultrafilter since $\{0,2,4,\ldots\} \notin \mathcal{F}^{co}$ (why?) and also $\mathbb{N} \setminus \{0,2,4,\ldots\} = \{1,3,5,\ldots\} \notin \mathcal{F}^{co}$ (why?)

Ex: Let $i\in I$. The set $\mathcal{F}^i=\{A\subseteq I : i\in A\}$ is an ultrafilter.

Pf: $i\notin\emptyset \Rightarrow \emptyset\notin\mathcal{F}^i$ (Ax 1)

$A,B\in\mathcal{F}^i \to i\in A\cap B \Rightarrow A\cap B\in \mathcal{F}^i$ (Ax 2)

$A\subseteq C\subseteq I \Rightarrow i\in C$, so $C\in\mathcal{F}^i$ (Ax 3)

if $A\in\mathcal{F}^i$, then $I\setminus A\notin \mathcal{F}^i$ b/c $i\in A\to i\notin I\setminus A$

~~and like ie~~ AND if $B\notin\mathcal{F}^i$ then $I\setminus B\in\mathcal{F}^i$ b/c $i\notin B\to i\in I\setminus B$ (Ax 4)

Def: We call $\mathcal{F}^i$ the <u>principal ultrafilter generated by i</u>.

FACT: Principal ultrafilters won't help us define "almost all."
— we need "nonprincipal ultrafilters" — their existence can be shown w/ Axiom of Choice..., cannot write them down!!

FACT: $\mathcal{F}^{co}$ is a subset of any non-principal ultrafilter $\mathcal{F}$ $\Rightarrow$ no finite set exists in $\mathcal{F}$

Def: Let $\mathcal{F}$ be nonprinc. ultrafilter over $\mathbb{N}$. We say $x\in\mathbb{R}^{\mathbb{N}}$ <u>has prop P almost everywhere</u> if $\{n\in\mathbb{N}: x_n \text{ has prop } P\}\in\mathcal{F}$.

(means the prop is true in ∞-many components of x)

Ex — we say that $r=(0,1,1,\dots)$ "equals 1 a.e." because

$r_1 \quad r_2 \quad r_3$

$\{n\in\mathbb{N}: r_n=1\}=\{2,3,4,\dots\}$ is infinite $\in\mathcal{F}$,

since $\{2,3,4,\dots\}\in\mathcal{F}^{co}\subseteq\mathcal{F}$.

Notation: $\{n\in\mathbb{N}: x_n \text{ has prop } P\}=\|P(x)\|$

Ex: Let us describe a positive infinitesimal. Let $\epsilon\in\mathbb{R}^{\mathbb{N}}$ be $\epsilon=(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots)$

Then $\forall r\in\mathbb{R}, r>0$, we express $r=(r,r,r,\dots)\in\mathbb{R}^{\mathbb{N}}$ and we can say

$\|\epsilon<r\|=\{n\in\mathbb{N}:\epsilon_n<r_n\}$

$=\{m,m+1,\dots\}$, where $\epsilon_m=\frac{1}{m}$ is first fraction $<r$.

Since $\forall r\in\mathbb{R}_{r>0}\ \exists m$ of this form, we see that $\|\epsilon<r\|\in\mathcal{F}$ for all $r\in\mathbb{R}, r>0$.

Ex: $\omega = (1,2,3,\dots) \in \mathbb{R}^{\mathbb{N}}$ is an inf # b/c $\forall r \in \mathbb{R}$

$\|\omega > r\| \in \mathcal{F}$, In fact, $\omega = \frac{1}{\varepsilon}$.

Some problems

Notice if $x = (0,1,1,\dots)$ and $y = (1,0,1,1,\dots)$, then

$$\|x=y\| = \{3,4,5,6,\dots\} \in \mathcal{F}$$

This ~~cases~~ is similar to how $\frac{1}{2}$ and $\frac{2}{4}$ represent "same" number.
We write $x \equiv y$ when $\|x=y\|$.
We fix this w/ equiv relation: define

$[x] = \{z \in \mathbb{R}^{\mathbb{N}} : z \equiv x\}$ and then the hyperreal #'s are

the set ${}^*\mathbb{R} = \{[x] : x \in \mathbb{R}^{\mathbb{N}}\}$.

Props of ${}^*\mathbb{R}$

1. ${}^*\mathbb{R}$ is a field (can add, subtrg mult, divide by nonzero elts)
2. (Extensions) if $A \subseteq \mathbb{R}$ then $\exists\ {}^*A \subseteq {}^*\mathbb{R}$ called the (unique) extension of A and all $f: \mathbb{R} \to \mathbb{R}$ extend to (unique) ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$

Def: (3) ("inf close") we say $x, y \in {}^*\mathbb{R}$ are inf. close provided that $x-y$ is infinitesimal

3. if $x \in {}^*\mathbb{R}$ is finite, then $\exists$ exactly one $y \in \mathbb{R}$ so that ~~$x-y$~~ y is inf close to x
   we say here that $y = st(x)$

4. (Transfer) Sentences using quantifiers ~~of~~ over $\mathbb{R}$ (not over subsets of $\mathbb{R}$) can be extended to sentences about ${}^*\mathbb{R}$.

Ex: Let $A = [0,1]$.

$$\exists u \in \mathbb{R}\ \forall z \in \mathbb{R}\ (\forall x \in A\ (z \geq x) \to u \leq z) \quad (u \text{ is "least upper bound" of } A)$$

$\Downarrow$ transfer

$$\exists u \in {}^*\mathbb{R}\ \forall z \in {}^*\mathbb{R}\ (\forall x \in {}^*A\ (z \geq x) \to u \leq z)$$

However

EX: $(\forall A \subset \mathbb{R}) \exists u \in \mathbb{R} \forall z \in \mathbb{R} \left(\forall x \in A, z \geq x \rightarrow u \leq z \right)$

DOES NOT transfer b/c of 2nd order logic appearing