21 March 2019 and 26 March 2019

Who shaves the barber?

Last class: imagine a town with precisely one barber. The barber obeys the
Logical Barber Property: "the barber shaves *all* the people, and only the people, who do not shave themselves."

Who shaves the barber?

Case 1: the barber does not shave himself
This is impossible: in this case, the **logical barber property** implies that the barber must shave himself (contrary to "case 1").
Case 2: the barber shaves himself
This is impossible: in this case, the barber shaves himself, but the **logical barber property** says that the barber shaves *all* people who do not shave themselves, so he must shave himself (contrary to "case 2").

A "naive" set theory

1.) **Axiom of Extensionality**
$(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y$
2.) **Axiom Schema of (Unrestricted) Comprehension**
For any (one-place) predicate P, we take the following as an axiom:
$(\exists z)(\forall x)(x \in z \leftrightarrow Px)$

A "naive" set theory

Consider the predicate P where Px stands in for $\neg(x \in x)$.
Consider the following proof (called "Russell's paradox")

$\{\mathrm{Ax}\,2\}$	(1) $(\exists z)(\forall x)(x \in z \leftrightarrow Px)$	Axiom 2
$\{\mathrm{Ax}\,2\}$	(2) $(\forall x)(x \in \alpha \leftrightarrow Px)$	1 ES
$\{\mathrm{Ax}\,2\}$	(3) $\alpha \in \alpha \leftrightarrow \neg(\alpha \in \alpha)$	2 US & Def of P
$\{\mathrm{Ax}\,2\}$	(4) $\alpha \in \alpha \rightarrow \neg(\alpha \in \alpha)$	3 Bicond, Simpl
$\{\mathrm{Ax}\,2\}$	(5) $\neg(\alpha \in \alpha) \rightarrow \alpha \in \alpha$	3 Bicond, Comm, Simpl
$\{4\}$	(6) $\alpha \in \alpha$	Premise (for contradiction)
$\{6, \mathrm{Ax}\,2\}$	(7) $\neg(\alpha \in \alpha)$	4 6 Detachment
$\{6, \mathrm{Ax}\,2\}$	(8) $(\alpha \in \alpha) \wedge \neg(\alpha \in \alpha)$	6 7 Adjunction
$\{\mathrm{Ax}\,2\}$	(9) $\neg(\alpha \in \alpha)$	6 8 RAA
$\{\mathrm{Ax}\,2\}$	(10) $\alpha \in \alpha$	5 9 Detachment
$\{\mathrm{Ax}\,2\}$	(11) $\neg(\alpha \in \alpha) \wedge (\alpha \in \alpha)$	9 10 Adjunction

This shows that Axiom 2 implies a contradiction... therefore we must reject it! This is what makes naive set theory "naive".

Zermelo set theory

Naive set theory is broken: one way to fix it is to "build up" sets from scratch and only allow sets to contain other sets which were created "a lower level" of construction.

Axiom 1 is the same as in naive set theory. Axioms 2, 4, and 5 give us ways to make new sets. Axiom 3 is a "restricted" form of naive set theory's Axiom 2. Axiom 7 explicitly forbids Russel's paradox.

1. two sets have same member if and only if they are the same set
2. we may always combine two sets into a new set
3. if P is a predicate and A is an already existing set, then the set of members x of A, for which Px is true, form a set
4. for any set, the "union" of all elements of the set forms a set
5. for any set x, there is a set y consisting of all subsets of x
6. there is an infinite set
7. sets cannot contain themselves

Generic set theory notation

We use "braces" "{" and "}" to denote sets. The empty set is precisely $\emptyset = \{\}$. If a set contains things, it is common to write those thing between the braces; for example the set containing a and b is written $\{a, b\}$.

Formally, to write $y = \{a, b\}$ means $(\exists y)(\forall x)(x \in y \leftrightarrow (x = a \vee x = b))$.

Recall the subset notation: $a \subseteq b$ abbreviates $(\forall z)(z \in x \rightarrow z \in y)$. Other abbreviations: $x \cup y = \{z \colon z \in x \wedge z \in y\}$ and $x \cap y = \{z \colon z \in x \wedge z \in y\}$

Zermelo set theory

0. **"Empty set"** $(\exists x)(x = \emptyset)$
1. **"Extensionality"**: $(\forall x)(\forall y)((\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y)$
2. **"Pairing"**: $(\forall w)(\forall z)(\exists y)(\forall x)(x \in y \leftrightarrow (x = w \vee x = z))$
3. **"Restricted Comprehension"**: For any predicate P and already existing set A,
   $(\exists z)(\forall x)(x \in z \leftrightarrow (x \in A \wedge Px))$
4. **"Union"**: $(\forall z)(\exists a)(\forall y)(\forall x)(x \in y \wedge y \in z \rightarrow x \in a)$
5. **"Power set"**: $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subseteq x)$
6. **"Infinity"**: $(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \rightarrow \{y \cup \{y\}\} \in x))$
7. **"Foundation"**: $(\forall s)(\neg(s = \emptyset) \vee \rightarrow (\exists x)(x \in s \wedge s \cap x = \emptyset))$

A tautological equivalence to make life easier

Let's call the following tautological equivalence "self or".

P	$P \vee P$	$P \vee P \leftrightarrow P$
T	T	T
F	F	T

Zermelo set theory

Prove that the set $\{\emptyset\}$ exists, i.e. prove $(\exists y)(\forall x)(x \in y \leftrightarrow x = \emptyset)$.

$\{\mathrm{Ax}\,2\}$	(1) $(\forall z)(\exists y)(\forall x)(x \in y \leftrightarrow (x = \emptyset \vee x = z))$	Axiom 2 US
$\{\mathrm{Ax}\,2\}$	(2) $(\exists y)(\forall x)(x \in y \leftrightarrow (x = \emptyset \vee x = \emptyset))$	1 US
$\{\mathrm{Ax}\,2\}$	(3) $(\exists y)(\forall x)(x \in y \leftrightarrow x = \emptyset)$	2 TE (self or)

note: we could also express this set as $\{\{\}\}$.

Ordered pairs

An *ordered pair* consisting of the sets x (first) and y (second) is described by the notation $(x, y) = \{x, \{x, y\}\}$. More formally, to say that $\{x, y\}$ exists means

$$(\exists z)(\forall a)(a \in z \leftrightarrow a = x \vee a = y).$$

And to say that $\{x, \{x, y\}\}$ exists means

$$(\exists z)(\forall a)(a \in z \leftrightarrow a = x \vee a = \{x, y\})$$

For example,

$$(\emptyset, \emptyset) \overset{\text{def}}{=} \{\emptyset, \{\emptyset, \emptyset\}\} \overset{\text{Ax.1}}{=} \{\emptyset, \{\emptyset\}\}$$

A *relation* is a set of ordered pairs; i.e. a relation is a set r such

that

$$(\forall x)(x \in r \rightarrow (\exists a)(\exists b)(x = \{a, \{a, b\}\})$$

A *function* is a relation R with the property that if $(a,b) \in R$ and $(a,c) \in R$, then $b=c$. In other words, if f is a function, then f is a relation such that

$$(\forall a)(\forall b)(\forall c)((a,b) \in f \wedge (a,c) \in f \to b = c)$$

note: we often abbreviate $(a,b) \in f$ as "$f(a) = b$"

Example:
a.) Consider the relation $r = \{(a,b),(a,c),(a,a)\}$. Is it a function?
b.) Consider the relation $r = \{(a,b),(b,b),(c,d)\}$. Is it a function?
c.) Is $r = \emptyset$ a relation? Is it a function?

Zermelo-Fraenkel set theory is Zermelo set theory along with the following axiom:

8. **"Replacement"** Let f be a function. Then for any x, the following is a set:

$$y = f[x] \stackrel{\text{def}}{=} \{f(z) \colon z \in x\}.$$

We will not worry about a formal version of replacement here...

Well orderings

A *well-order* of a set x is a total order (i.e. it obeys the axioms of total order theory) with the additional property that all subsets of x have a smallest element.

Examples:
a.) Is $\mathbb{N} = \{0, 1, 2, \ldots\}$ with the usual "$\leq$" well ordered?
b.) Is the set of real numbers $\mathbb{R}$ with the usual "$\leq$" well ordered?
c.) Is $\emptyset$ well-ordered?

Russell: To select one sock from each of infinitely many pairs of socks requires the Axiom of Choice; but for shoes the Axiom is not needed.

The following axiom is called the axiom of choice:

9. **"Choice"** For all sets x, there is a relation r which is a well ordering of x.

Relationships between the set theories

Denote the set theory defined by axioms 1-8 "ZF" and denote the set theory defined by axioms 1-9 "ZFC".

In ZFC...

- Banach-Tarski paradox
- every "connected graph" has a "spanning tree"
- a "group" exists on any set
- "non-measurable sets" exist
- every "Hilbert space" has an "orthonormal basis"

In ZF....

- the real numbers can be partitioned into strictly more sets than there are real numbers
- there is a tree T with no leaves, but which has no infinite path
- there is a field with no "algebraic closure"...moreover there are more algebraic closures for $\mathbb{Q}$ than $\mathbb{C}$
- there is a vector spaces without a basis

Some interesting reads

- A Peculiar Connection Between the Axiom of Choice and Predicting the Future
- Set theory and weather prediction