# Sets

A common theme in mathematics in general: "group together the items of discussion into an object/word/symbol so that you can talk about them all simultaneously".

**Examples:**

- "the real numbers" $\mathbb{R}$, "the integers" $\mathbb{Z}$, "surreal numbers"
- "even numbers", "prime numbers", "vampire numbers"
- "the continuous functions", "the differentiable functions", "the locally integrable functions"
- "Khinchin's constant", "Mills' constant", "the number $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$"
- "all objects $x$ in the universe with the property that the predicate $Px$ is true"

---

**Sets** are the formal objects we are describing here.

# Sets

*Sets* are an "unordered collection" of objects. This distinguishes a set from a sequence:
$a_n = (-1)^n$ "is" $1, -1, 1, -1, 1, \ldots$, which is an "ordered collection" of *members* of the set $\{-1, 0\}$.

We would like to have $\{-1, -1, 0\} = \{-1, 0\}$ to reflect that sets do not have "repeat" members ("multisets" allow this!).

Natural idea: let's try to make a "set theory".

# Sets

What properties do “unordered collections” have? Fundamentally:

1. two sets have the same members if and only if they are the same set
2. given any predicate $P$, the set of $x$ in the universe such that $Px$ is true should be a set

(note: we can't write $(\forall P)$ where $P$ is a predicate in first order logic... we “cheat” and use an “axiom schema”, meaning we accept the axiom for any $P$ we need in a given proof)

# A "naive" set theory

This theory has a two term predicate $\in$ ($x \in y$ is pronounced "$x$ is an element of $y$") along with the following axioms:

1.) **Axiom of Extensionality**

$(\forall z)(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$

2.) **Axiom Schema of (Unrestricted) Comprehension**

For any (one-place) predicate $P$, we take the following as an axiom:

$(\exists z)(\forall x)(x \in z \leftrightarrow Px)$

---

note: "schema" refers to the idea that we are allowing this as an axiom for every predicate $P$ of first order logic (either all at once, or "as needed") – we may **never** write $(\forall P)$, where $P$ is a predicate, in first order logic!

# A "naive" set theory

An "abbreviation": we *define* the symbols "$x \subseteq y$" as an abbreviation for "$(\forall z)(z \in x \to z \in y)$". We may cite this definition in order to replace abbreviated symbols.

In other words, $x \subseteq y = (\forall z)(z \in x \to z \in y)$. Now we can prove a simple theorem: $(x \subseteq y) \wedge (y \subseteq x) \to x = y$.

# A "naive" set theory

Prove $(\exists z)(\forall x)(\neg(x \in z))$. Let $Px$ be a predicate that denotes "$\neg(x = x)$". Then prove

| | | |
|---|---|---|
| {Ax 2} | (1) $(\exists z)(\forall x)(x \in z \leftrightarrow Px)$ | Axiom 2 |
| {Ax 2} | (2) $(\forall x)(x \in \alpha \leftrightarrow Px)$ | 1 ES |
| {Ax 2} | (3) $x \in \alpha \leftrightarrow \neg(x = x)$ | 2 US & Def of $P$ |
| {Ax 2} | (4) $x \in \alpha \to \neg(x = x)$ | 3 Bicond., Commutative, & Si |
| {Ax 2} | (5) $\neg(\neg(x = x)) \to \neg(x \in \alpha)$ | 4 Contraposition |
| {Ax 2} | (6) $x = x \to \neg(x \in \alpha)$ | 5 TE (Double negation) |
| {} | (7) $x = x$ | Identity law |
| {Ax 2} | (8) $\neg(x \in \alpha)$ | 6 7 Detachment |
| {Ax 2} | (9) $(\forall x)(\neg(x \in \alpha))$ | 8 UG |
| {Ax 2} | (10) $(\exists z)(\forall x)(\neg(x \in z))$ | 9 EG |

We give a special symbol for this important set we proved to exist: $\emptyset$ ("empty set")

**Empty Set Theorem**: $(\forall x)(\neg(x \in \emptyset))$

# A "naive" set theory

Prove $(x \subseteq y) \wedge (y \subseteq x) \rightarrow x = y$ in naive set theory:

| | | |
|---|---|---|
| $\{1\}$ | (1) $(x \subseteq y) \wedge (y \subseteq x)$ | Premise |
| $\{1\}$ | (2) $x \subseteq y$ | 1 Simplification |
| $\{1\}$ | (3) $(y \subseteq x) \wedge (x \subseteq y)$ | 1 Commutative law of $\wedge$ |
| $\{1\}$ | (4) $y \subseteq x$ | 3 Simplification |
| $\{1\}$ | (5) $(\forall z)(z \in x \rightarrow z \in y)$ | 2 Definition of $\subseteq$ |
| $\{1\}$ | (6) $(\forall z)(z \in y \rightarrow z \in x)$ | 4 Definition of $\subseteq$ |
| $\{1\}$ | (7) $z \in x \rightarrow z \in y$ | 5 US |
| $\{1\}$ | (8) $z \in y \rightarrow z \in x$ | 6 US |
| $\{1\}$ | (9) $z \in x \leftrightarrow z \in y$ | 7 8 Adjunction & Biconditi |
| $\{1\}$ | (10) $(\forall z)(z \in x \leftrightarrow z \in y)$ | 9 UG |
| $\{\mathrm{Ax}\,1\}$ | (11) $(\forall z)(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$ | Axiom of Extensionality |
| $\{\mathrm{A}.1\}$ | (12) $(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y$ | 11 Bicond., Comm. & Sim |
| $\{1, \mathrm{Ax}\,1\}$ | (13) $x = y$ | 10 12 Detachment |
| $\{\mathrm{Ax}\,1\}$ | (14) $(x \subseteq y) \wedge (y \subseteq x) \rightarrow x = y$ | 1 13 C.P. |

# A "naive" set theory

Prove $(\forall y)(\emptyset \subseteq y)$ in naive set theory. We will proceed with proof by contradiction:

| | | |
|---|---|---|
| $\{1\}$ | (1) $\neg(\forall y)(\emptyset \subseteq y)$ | Premise |
| $\{1\}$ | (2) $\neg\neg(\exists y)(\neg((\forall z)(z \in \emptyset \to z \in y)))$ | 1 Q & Def of $\subseteq$ |
| $\{1\}$ | (3) $(\exists y)(\neg\neg(\exists z)\neg(z \in \emptyset \to z \in y))$ | 2 Q & TE (Double negat |
| $\{1\}$ | (4) $(\exists y)(\exists z)(z \in \emptyset \wedge \neg(z \in y))$ | 3 TE(Neg.imp & Dbl neg |
| $\{1\}$ | (5) $(\exists z)(z \in \emptyset \wedge \neg(z \in \alpha))$ | 4 ES |
| $\{1\}$ | (6) $\beta \in \emptyset \wedge \neg(\beta \in \alpha)$ | 5 ES |
| $\{1\}$ | (7) $\beta \in \emptyset$ | 6 Simplification |
| $\{\mathrm{Ax}\,2\}$ | (8) $(\forall y)(\neg(y \in \emptyset))$ | **Empty Set Theorem** |
| $\{\mathrm{Ax}\,2\}$ | (9) $\neg(\beta \in \emptyset)$ | 8 US |
| $\{1, \mathrm{Ax}\,2\}$ | (10) $\beta \in \emptyset \wedge \neg(\beta \in \emptyset)$ | 7 9 Adjunction |
| $\{\mathrm{Ax}\,2\}$ | (11) $(\forall y)(\emptyset \subseteq y)$ | 1 10 R.A.A. |

# A "naive" set theory

Prove $(\exists z)(\emptyset \in z)$. Let $Px$ denote the predicate "$x = \emptyset$". Then prove

| | | |
|---|---|---|
| $\{\mathrm{Ax}\,2\}$ | (1) $(\exists z)(\forall x)(x \in z \leftrightarrow Px)$ | Axiom 2 |
| $\{\mathrm{Ax}\,2\}$ | (2) $(\forall x)(x \in \alpha \leftrightarrow Px)$ | 1 ES |
| $\{\mathrm{Ax}\,2\}$ | (3) $\emptyset \in \alpha \leftrightarrow \emptyset = \emptyset$ | 2 US & Def of $P$ |
| $\{\mathrm{Ax}\,2\}$ | (4) $\emptyset = \emptyset \rightarrow \emptyset \in \alpha$ | 3 Bicond. & Simplify |
| $\{\}$ | (5) $\emptyset = \emptyset$ | Identity Law |
| $\{\mathrm{Ax}\,2\}$ | (6) $\emptyset \in \alpha$ | 4 5 Detachment |
| $\{\mathrm{Ax}\,2\}$ | (7) $(\exists z)(\emptyset \in z)$ | 6 EG |

# The Liar Paradox

"This sentence is false."

Is the sentence true or is it false?

# The Barber paradox

Imagine a town with precisely one barber. The barber obeys the property "the barber shaves anyone who does not shave themselves".

Who in the town shaves the barber?