Homework 9 – MATH 2510 Spring 2019

Recall: an ordinal is a transitive well-ordered set. The first few ordinals are: $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{0,1\}, 3 = \{0,1,2\}, 4 = \{0,1,2,3\}, \ldots$. The first infinite ordinal is $\omega = \{0,1,2,3,\ldots\}$. We think of ordinals as being "ordered by ϵ", i.e. if $\alpha \in \beta$, then we think of $\alpha < \beta$. A function $f\colon A \to B$ is called one-to-one provided that whenever $f(x) = f(y)$, it follows that $x = y$. For sets A and B, we say that $\text{card}(A) \preceq \text{card}(B)$ provided there is a one-to-one function $f\colon A \to B$. A cardinal number is an ordinal number with the property that if $\beta < \alpha$ (ordinals) then it follows that $\text{card}(\beta) \preceq \text{card}(\alpha)$ (cardinals). Let α and β be cardinal numbers, we define cardinal addition by

$$\alpha \oplus \beta = \text{card}\left((\alpha \times \{0\}) \bigcup (\beta \times \{1\})\right).$$

Cardinal multiplication is defined by

$$\alpha \otimes \beta = \text{card}(\alpha \times \beta).$$

If A and B are sets, then ${}^{B}A$ denotes the set of functions whose domain is B and whose codomain is A.

$${}^{B}A = \left\{ f\colon B \to A \,\middle|\, f \text{ is a function} \right\}.$$

Cardinal exponentiation is defined by

$$\alpha^{\beta} = |{}^{\beta}\alpha|.$$

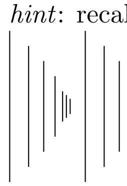
Consider an ordinal α – we say that a subset $\beta \subset \alpha$ is cofinal with α if it obeys the following property:

$$\forall a \in \alpha \exists b \in \beta (a \in b).$$

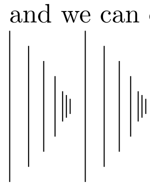
The cofinality of a set α is defined to be the minimum cardinality of all sets which are cofinal with α, i.e.

$$\text{cf}(\alpha) = \min\{|\beta| \colon \beta \text{ is cofinal with } \alpha\}.$$

1. Show that the ordinals $\text{card}(\omega + 3) = \text{card}(\omega + \omega)$ by demonstrating appropriate one-to-one functions.
   hint: recall that we can draw $\omega + 3$ in the following way:

   and we can draw $\omega + \omega$ in the following way:

2. Consider the ordinal number $\alpha = \{0,1,2,3\}$.

   (a) List all subsets of α (i.e. find the power set $\mathscr{P}(\alpha)$).

(b) Cross out all subsets of α that are **not cofinal** with α.

(c) Write down the cardinalities of all cofinal subsets of α.

(d) Write $\mathrm{cf}(\alpha)$.

(e) (Bonus): What is $\mathrm{cf}(\omega + 1)$? (*hint*: recall that $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$)

3. Calculate the requested cardinal arithmetic operation. Carefully do it by the definition (i.e. show your sets, etc).

   (a) $1 \oplus 2$

   (b) $3 \otimes 3$

   (c) 4^2