

Homework 8 – MATH 2510 Spring 2019

An ordered pair (a, b) is the set $\{a, \{a, b\}\}$. A relation is a set of ordered pairs. A function f is a relation with the property that if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$. We write $f: A \rightarrow B$ to mean that f is a function with domain (“where inputs live”) A and (“where outputs live”) codomain B . The range of a function is given by $\text{range}(f) = \{y: \exists x(x, y) \in f\}$. A one-to-one function is a function f with the property that if $(a, b) \in f$ and $(c, b) \in f$, then $a = c$ (“no output has two different inputs”). We say that $\text{card}(X) \preceq \text{card}(Y)$ if there is a one-to-one function $f: X \rightarrow Y$. We say that $\text{card}(X) = \text{card}(Y)$ whenever both $X \preceq Y$ and $Y \preceq X$.

1. Is the function $\begin{cases} f: \mathbb{N} \rightarrow \mathbb{N} \\ f(n) = 2n \end{cases}$ one-to-one? Draw/sketch it (software ok) and explain why or why not.
2. Is the function $\begin{cases} f: [-1, 1] \rightarrow \mathbb{R} \\ f(x) = (x + 1)^2 \end{cases}$ one-to-one? Draw/sketch it (software ok) and explain why or why not.
3. Is the function $\begin{cases} f: (0, \infty) \rightarrow \mathbb{R} \\ f(x) = x + \ln(x) \end{cases}$ one-to-one? Draw/sketch it (software ok) and explain why or why not.
4. Is the function $\begin{cases} f: \left(\frac{\pi}{2}, \pi\right) \rightarrow \mathbb{R} \\ f(x) = \sin\left(\frac{x^2 + 1}{x^3}\right) \end{cases}$ one-to-one? Draw/sketch it (software ok) and explain why or why not.
5. Consider the set \mathbb{P}_2 of polynomials of degree ≤ 2 and \mathbb{P}_1 , i.e. $\mathbb{P}_2 = \{ax^2 + bx + c: a, b, c \in \mathbb{R}\}$. Define the function

$$\begin{cases} D: \mathbb{P}_2 \rightarrow \mathbb{P}_2 \\ D(ax + b) = \left(\frac{d^2}{dx^2} + x\right)(ax^2 + bx + c) \\ = (ax^2 + bx + c)'' + x(ax^2 + bx + c). \end{cases}$$

Is this function one-to-one? Why or why not?

6. Define a function (using symbols or a picture) to show that $\text{card}(\{0, 2, 4, 6, 8, 10, \dots\}) \preceq \text{card}(\{0, 1, 2, 3, 4, 5, \dots\})$.
7. Define a function (using symbols or a picture) to show that $\text{card}(\mathbb{Z}) \preceq \text{card}(\mathbb{R})$.
8. Show that $\text{card}\left(\left(-2, 3\right)\right) \preceq \text{card}(\mathbb{R})$. (hint: *think about “rational functions”*). Write **formulas** for your functions and draw.
9. Show that $\text{card}\left(\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right) = \text{card}(\mathbb{R})$ (hint: *inverse trig functions*). Write **formulas** for your functions and draw.