HW7 MATH 2502 Spring 2019

#196 $\int \frac{1}{(x-3)(x-2)}dx =$

partial fractions $\rightarrow \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$

mult by $(x-3)(x-2)$ $\rightarrow 1 = A(x-2) + B(x-3)$

$0x+1 = (A+B)x + (-2A-3B)$

equate coeffs:

$$\begin{cases} A+B=0 \rightarrow \boxed{A=-B} \\ -2A-3B=1 \rightarrow -2(-B)-3B=1 \end{cases}$$

$-B=1$

$\boxed{B=-1}$

$\boxed{A=1}$

Therefore,

$$\int \frac{1}{(x-3)(x-2)}dx = \int \frac{1}{x-3}dx - \int \frac{1}{x-2}dx$$

$$= \ln(x-3) - \ln(x-2) + C$$

#197 $\int \frac{3x}{x^2+2x-8}dx$

partial fractions $\rightarrow \frac{3x}{x^2+2x-8} = \frac{3x}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$

Mult by $(x+4)(x-2)$

$\Rightarrow 3x = A(x-2) + B(x+4)$

$3x + 0 = (A+B)x + (-2A+4B)$

equate coefficients:

$$\begin{cases} A+B=3 \longrightarrow \boxed{A=3-B} \\ -2A+4B=0 \longrightarrow -2(3-B)+4B=0 \\ \qquad -6+2B+4B=0 \\ \qquad 6B=6 \longrightarrow \boxed{B=1} \end{cases} \qquad \boxed{A=3-1=2}$$

Therefore,

$$\int \frac{3x}{x^2+2x-8}\,dx = \int \frac{2}{x+4}\,dx + \int \frac{1}{x-2}\,dx$$

$$= 2\ln(x+4) + \ln(x-2) + C$$

#198 $\displaystyle\int \frac{1}{x^3-x}\,dx$

Partial fractions

$$\frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

Multiply by $x(x-1)(x+1)$ to get

$$1 = A\underbrace{(x-1)(x+1)}_{x^2-1} + B\underbrace{x(x+1)}_{x^2+x} + C\underbrace{x(x-1)}_{x^2-x}$$

$$0x^2+0x+1 = (A+B+C)x^2 + (B-C)x + (-A)$$

$$\Rightarrow \begin{cases} A+B+C=0 \\ B-C=0 \rightarrow \boxed{B=C} \rightarrow -1+B+B=0 \\ -A=1 \rightarrow \boxed{A=-1} \end{cases}$$

$$2B=1$$

$$\boxed{B=1/2}$$

$$\boxed{C=1/2}$$

Therefore,

$$\int \frac{1}{x^3-x}\,dx = -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= -\ln(x) + \frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) + C$$

Section 3.7

#356 $\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx$

$\boxed{\begin{array}{l} u=\ln(x) \\ du=\frac{1}{x}dx \end{array}}$

$$\hookrightarrow = \lim_{b\to\infty} \int_0^{\ln(b)} u\,du$$

$$= \lim_{b\to\infty} \frac{u^2}{2}\Big|_0^{\ln(b)}$$

$$= \lim_{b\to\infty} \frac{1}{2}(\ln(b))^2 - 0$$

$= \infty \rightarrow$ integral diverges

#358 $\int_0^1 \ln(x)\,dx = \lim_{a\to 0^+} \int_a^1 \ln(x)\,dx$

$= \lim_{a\to 0^+} \left[x\ln(x) - x \right]_a^1$

$= \lim_{a\to 0^+} \left[(1\cancel{\ln(1)}^{0} - 1) - (a\ln(a) - a) \right]$

$= \lim_{a\to 0^+} (-1 - \cancel{a\ln(a)}^{0} - \cancel{a}^{0})$ since log << polynomial ⟹ "polynomial wins the race"

$= -1$

#362 $\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$

$= \lim_{b\to\infty} \left[-e^{-x} \right]_0^b$

$= \lim_{b\to\infty} \left[-\cancel{e^{-b}}^{0} - (-e^0) \right]$

$= 1$

#366 $\int_0^2 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} \int_a^2 x^{-2}\,dx$

$= \lim_{a\to 0^+} \left. \frac{x^{-1}}{-1} \right|_a^2$

$= \lim_{a\to 0^+} \frac{2^{-1}}{-1} - \frac{a^{-1}}{-1}$

$= \lim_{a\to 0^+} \cancel{\frac{1}{a}}^{\infty} - \frac{1}{2} = \infty \rightarrow$ integral diverges

#370 $\displaystyle\int_1^\infty \frac{5}{x^3}\,dx = 5\lim_{b\to\infty}\int_1^b x^{-3}\,dx$

$\displaystyle = 5\left(\lim_{b\to\infty} \frac{x^{-2}}{-2}\Big|_1^b\right.$

$\displaystyle = \frac{-5}{2}\lim_{b\to\infty}\left(\cancel{\frac{1}{b^2}}^{\,0} - 1\right)$

$\displaystyle = \frac{5}{2}$

#374 $\displaystyle\int_1^\infty \frac{1}{x^e}\,dx = \lim_{b\to\infty}\int_1^b x^{-e}\,dx$

$\displaystyle = \lim_{b\to\infty} \frac{x^{-e+1}}{-e+1}\Big|_1^b$

$\displaystyle = \lim_{b\to\infty} \frac{\cancel{b^{-e+1}} - 1}{1-e} \to 0$

since $-e+1<0$

$-e+1 \approx -2.7+1$
$= -1.7,$
so $\displaystyle b^{-e+1} = \frac{1}{b^{|1-e|}}$

$\displaystyle = \frac{-1}{1-e} = \frac{1}{e-1}$

$u = x \quad dv = e^{-x}dx$
$du = dx \quad v = -e^{-x}$

$\boxed{\#382}$ $\displaystyle\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$

(parts) $\displaystyle = \lim_{b\to\infty}\left(-xe^{-x}\Big|_0^b + \underbrace{\int_0^b e^{-x}\,dx}_{-e^{-x}\big|_0^b}\right)$

0 b/c exponential beats polynom

$\displaystyle = \lim_{b\to\infty}\left[\left(\cancel{-be^{-b}}+0\right)+\left(\cancel{-e^{-b}}+e^0\right)\right]$

$= 1$

$\boxed{\text{Problem A}}$ $\displaystyle\int \frac{2x+1}{x^2+4x+4}\,dx$

partial fractions

$$\frac{2x+1}{x^2+4x+4} = \frac{2x+1}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$\downarrow$ multiply by $(x+2)^2$

$2x+1 = A(x+2)+B$

$2x+1 = Ax + (2A+B)$

$\begin{cases} \boxed{A=2} \to 2(2)+B=1 \\ 2A+B=1 \to \boxed{B=-3} \end{cases}$

Therefore,

$$\int\frac{2x+1}{x^2+4x+4}\,dx = 2\int\frac{1}{x+2}\,dx - 3\int\frac{1}{(x+1)^2}\,dx$$

$$= 2\ln(x+2) + \frac{3}{x+1} + C$$

Problem B | $\int \frac{x-1}{(x+1)(x^2+9)}\,dx$

Partial fractions

$$\frac{x-1}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$\Big\downarrow \text{ mult by } (x+1)(x^2+9)$$

$$x-1 = A(x^2+9) + \underbrace{(Bx+C)(x+1)}_{Bx^2+(B+C)x+C}$$

$$0x^2+x-1 = (A+B)x^2 + (B+C)x + (9A+C)$$

$$\begin{cases} A+B=0 \longrightarrow \boxed{A=-B} \\ B+C=1 \longrightarrow \boxed{C=1-B} \\ 9A+C=-1 \longrightarrow 9(-B)+(1-B)=-1 \end{cases}$$

$$-10B=-2$$

$$\boxed{B=\frac{1}{5}} \quad \boxed{A=-1/5}$$

$$\boxed{C=1-\frac{1}{5}=4/5}$$

Therefore,

$$\int \frac{x-1}{(x+1)(x^2+9)}\,dx = -\frac{1}{5}\int \frac{1}{x+1}\,dx + \frac{1}{5}\underbrace{\int \frac{x}{x^2+9}\,dx}_{u=x^2+9 \quad \frac{1}{2}du=x\,dx} + \frac{4}{5}\underbrace{\int \frac{1}{x^2+9}\,dx}$$

$$\frac{1}{10}\int \frac{1}{u}\,du$$

$$= \int \frac{1/9}{\left(\frac{x}{3}\right)^2+1}\,dx \qquad u=\frac{x}{9} \qquad 9du=dx$$

$$= \frac{1}{3}\int \frac{1}{u^2+1}\,du$$

$$= -\frac{1}{5}\ln(x+1) + \frac{1}{10}\ln(x^2+9) + \frac{4}{15}\arctan\left(\frac{x}{3}\right) + C$$

Problem 6 $\displaystyle\int \frac{1}{(x^2+36)(x^2+4)}\,dx$

$$\frac{1}{(x^2+36)(x^2+4)} = \frac{Ax+B}{x^2+36} + \frac{Cx+D}{x^2+4}$$

$\downarrow$ mult by $(x^2+36)(x^2+4)$

$$1 = \underbrace{(Ax+B)(x^2+4)}_{\cancel{Ax^3+4Ax+Bx^2+4B}} + \underbrace{(Cx+D)(x^2+36)}_{\cancel{Cx^3+36Cx+Dx^2+36D}}$$

$$0x^3+0x^2+0x+1 = (A+C)x^3+(B+D)x^2+(4A+36C)x+(4B+36D)$$

$$\begin{cases} A+C=0 \rightarrow \boxed{A=-C} \\ B+D=0 \rightarrow \boxed{B=-D} \\ 4A+36C=0 \rightarrow -4C+36C=0 \rightarrow 32C=0 \rightarrow \boxed{C=0} \quad \boxed{A=0} \\ 4B+36D=1 \end{cases}$$

$$-4D+36D=1$$

$$32D=1$$

$$\boxed{D=1/32} \quad \boxed{B=-1/32}$$

Therefore,

$$\int \frac{1}{(x^2+36)(x^2+4)}\,dx = -\frac{1}{32}\int \frac{1}{x^2+36}\,dx + \frac{1}{32}\int \frac{1}{x^2+4}\,dx$$

$$= -\frac{6}{32}\arctan\left(\frac{x}{6}\right) + \frac{2}{32}\arctan\left(\frac{x}{2}\right) + C$$