§6.1

#13 $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$ ← centered at 0

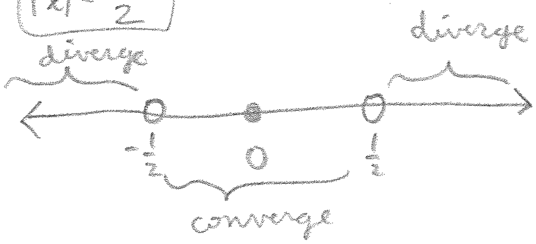
Soln: Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{n+1}}{\frac{(2x)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{(2x)^{n+1}\, n}{(2x)^n (n+1)} \right|$$

$$= \lim_{n\to\infty} 2|x| \frac{n}{n+1} \nearrow 1$$

$$= 2|x| \underset{\substack{\uparrow \\ \text{need}}}{<} 1$$

$$\Rightarrow \boxed{|x| < \frac{1}{2}}$$



endpoints

@ $+\frac{1}{2}$

Plug into series:

$$\sum_{n=1}^{\infty} \frac{(2(\frac{1}{2}))^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

harmonic series... diverge

$-\frac{1}{2}$

Plug into series:

$$\sum_{n=1}^{\infty} \frac{(2(-\frac{1}{2}))^n}{n}$$

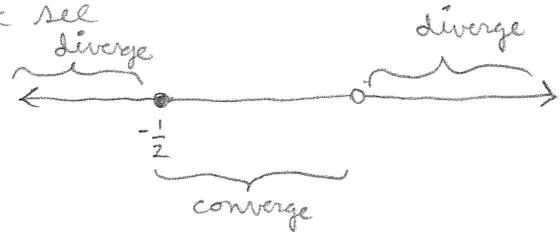
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Alternating series... use AST:

$\lim \frac{1}{n} = 0$

$\Rightarrow$ converges

So we see



ROC $= \frac{1}{2}$

IOC $= [-\frac{1}{2}, \frac{1}{2})$

#15 $\displaystyle\sum_{n=1}^{\infty} \frac{n x^n}{2^n} \leftarrow$ centered at 0

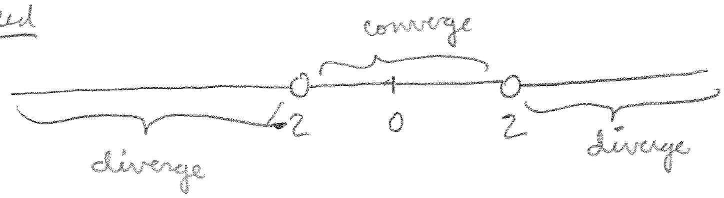
Soln: Ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)x^{n+1}}{2^{n+1}}}{\frac{n x^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}\, 2^n}{2^{n+1}\, n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{2} \frac{(n+1)}{n} \right|$$

$$= \frac{|x|}{2} \underset{\text{need}}{\uparrow}< 1$$

$\Rightarrow |x| < 2$



endpoints

−2

Plug in:

$$\sum_{n=1}^{\infty} \frac{n(-2)^n}{2^n}$$

$$= \sum_{n=1}^{\infty} (-1)^n n$$

diverges by test for divergence

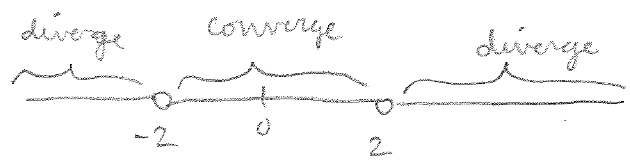
2

Plug in:

$$\sum_{n=1}^{\infty} \frac{n\, 2^n}{2^n} = \sum_{n=1}^{\infty} n$$

diverges by test for divergence

$\Longrightarrow$



ROC = 2

IOC = $(-2, 2)$

#18 $\sum_{K=1}^{\infty} \frac{k^e x^k}{e^k}$ ← centred at 0

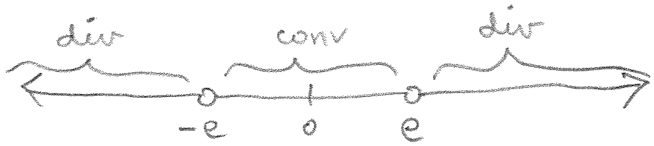
Soln: By ratio test,

$$\lim_{K\to\infty} \left| \frac{\frac{(k+1)^e x^{k+1}}{e^{k+1}}}{\frac{k^e x^k}{e^k}} \right| = \lim_{K\to\infty} \frac{(k+1)^e |x|^{\cancel{k+1}} \cancel{e^k}}{e^{\cancel{k+1}} k^e \cancel{|x|^k}}$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{\overset{1}{\nearrow} e} \cdot \frac{1}{e}|x|$$

$$= \frac{|x|}{e} \underset{\text{need}}{<} 1$$

$\Rightarrow |x| < e$



endpts

e

Plug in:

$$\sum_{K=1}^{\infty} \frac{k^e e^k}{e^k} = \sum_{K=1}^{\infty} k^e$$

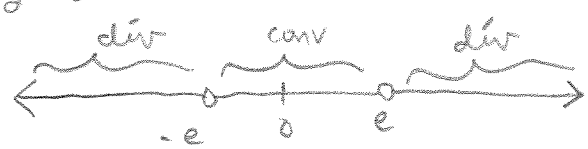
diverges by TFD

-e

plug in

$$\sum_{K=1} \frac{k^e(-e)^k}{e^k} = \sum_{K=1}^{\infty} (-1)^K k^e$$

diverges by TFD



ROC = e

IOC = (-e, e)

$\boxed{\#24}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!\,x^n}{n^{2n}}$

Soln: Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(2(n+1))!\,x^{n+1}}{(n+1)^{2(n+1)}}}{\frac{(2n)!\,x^n}{n^{2n}}} \right| = \lim_{n\to\infty} \frac{(2n+2)!\,|x|^{n+1}\, n^{2n}}{(n+1)^{2n+2}\,(2n)!\,|x|^n}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} \frac{n^{2n}}{(n+1)^2 (n+1)^{2n}} |x|$$

$$= |x| \lim_{n\to\infty} \underbrace{\frac{(2n+2)(2n+1)}{(n+1)^2}}_{= \frac{4n^2+\cdots}{n^2+\cdots} \xrightarrow{n\to\infty} 4} \underbrace{\left(\frac{n}{n+1}\right)^{2n}}_{= e^{2n \ln\left(\frac{n}{n+1}\right)}}$$

But $\infty \cdot 0$

$$\lim 2n \ln\left(\frac{n}{n+1}\right)$$

$$= \lim \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{2n}}$$

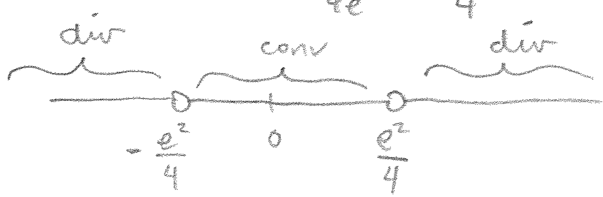
$$\overset{L'H}{=} \lim \frac{\left(\frac{1}{\frac{n}{n+1}}\right)\left(\frac{(n+1)(1) - n(1)}{(n+1)^2}\right)}{-\frac{1}{2n^2}}$$

$$= \lim \frac{(n+1)(-2n^2)}{n(n+1)^2} = \lim \frac{-2n^3 - 2n^2}{n^3 + \cdots}$$

$$= -2$$

$$= |x| 4 e^{-2} \underset{\text{need}}{<} 1$$

$$\Rightarrow |x| < \frac{1}{4e^{-2}} = \frac{e^2}{4}$$



$\Rightarrow$ ROC is $\frac{e^2}{4}$

§6.2

$\boxed{\#87}$ From

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Big\downarrow \text{ replace } x \text{ with } -x$$

$$\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}$$

$\boxed{\#90}$ From

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Big\downarrow \text{ replace } x \text{ w/ } -x$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

$$\Big\downarrow \frac{d}{dx}$$

$$\sum_{n=0}^{\infty} n(-1)^n x^{n-1} = \frac{-1}{(1+x)^2}$$

$$\Big\downarrow \text{ multiply } -2x$$

$$-2\sum_{n=0}^{\infty} n(-1)^n x^n = \frac{2x}{(1+x)^2}$$

$\boxed{\#92}$

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Big\downarrow \frac{d}{dx}$$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

$$\Big\downarrow \text{ plug in } x = \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{n}{3^{n-1}} = \frac{1}{(1-\frac{1}{3})^2} = \frac{1}{(2/3)^2}$$

$$\Big\downarrow \text{ multiply by } \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \left(\frac{9}{4}\right)\left(\frac{1}{3}\right)$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{3}{4}$$

Section 6.3

$\boxed{\#119}$ $f(x) = \sin(2x)$ centered at $a = \frac{\pi}{2}$

Taylor poly of degree 2 centered at $a = \frac{\pi}{2}$:

$$f\left(\tfrac{\pi}{2}\right) + f'\left(\tfrac{\pi}{2}\right)\left(x - \tfrac{\pi}{2}\right) + f''\left(\tfrac{\pi}{2}\right)\left(x - \tfrac{\pi}{2}\right)^2$$

$f(\pi/2) = \sin(\pi) = 0$

$f'(x) = 2\cos(2x) \rightsquigarrow f'\left(\frac{\pi}{2}\right) = 2\cos(\pi) = -2$

$f''(x) = -4\sin(2x) \longrightarrow f''\left(\frac{\pi}{2}\right) = -4\sin\left(2\left(\frac{\pi}{2}\right)\right) = -4\sin(\pi) = 0$

$$\Longrightarrow \quad 0 + (-2)\left(x - \tfrac{\pi}{2}\right) + 0$$

$\boxed{\#120}$ $f(x) = \sqrt{x}$

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	$\sqrt{4} = 2$
1	$\frac{1}{2\sqrt{x}}$	$\frac{1}{2\sqrt{4}} = \frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4} \cdot \frac{1}{(\sqrt{2})^3} = -\frac{1}{8\sqrt{2}}$

$\frac{1}{2}x^{-1/2}$

$$2 + \tfrac{1}{4}(x-4) - \tfrac{1}{8\sqrt{2}}(x-4)^2$$