HW 10 MATH 2502 Spring 2019

§5.4 #197 | Here $a_n = \frac{1}{2n-1}$ so take $b_n = \frac{1}{n}$. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} > 0,$$

so by limit comparison test, $\sum a_n$ and $\sum b_n$ do same.

But $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series w/ p=1... harmonic series)

Thus $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges

#198 | Here, $a_n = \frac{1}{(n\ln(n))^2}$, so pick $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(n\ln(n))^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2(\ln(n))^2}$$

$$= \lim_{n\to\infty} \frac{1}{(\ln(n))^2} = 0$$

And $\sum_{n=2}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, p=2)

Therefore $\sum a_n = \sum_{n=2}^{\infty} \frac{1}{(n\ln(n))^2}$ converges

#203) Here $a_n = \frac{\sin(\frac{1}{n})}{\sqrt{n}}$. Taking $b_n = \frac{1}{\sqrt{n}}$ does not work w/ limit comparison test (try + see)

so try $b_n = \frac{1}{n^{3/2}}$:

need to use L'Hôpital

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sin(\frac{1}{n})}{\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \underbrace{n\sin\left(\frac{1}{n}\right)}_{\infty \cdot 0}$$

rewrite for L'Hôp

$$= \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}}$$

chain rule

$$\stackrel{L'H}{=} \lim_{n\to\infty} \frac{\cos(\frac{1}{n})\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}}$$

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

And $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series w/ $p = 3/2$)

Therefore, $\sum a_n = \sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}}$ converges as well

#211) Here $a_n = \frac{1}{4^n - 3^n}$ so take $b_n = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$.

Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{4^n-3^n}}{\frac{1}{4^n}} = \lim_{n\to\infty} \frac{4^n}{4^n - 3^n} = 1$$

Since $\sum b_n = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$ converges (geometric, $|r|<1$),

we conclude that

$$\sum a_n = \sum_{n=1}^{\infty} \frac{1}{4^n - 3^n} \text{ converges as well}$$

#212 Here $a_n = \frac{1}{n^2 - n\sin(n)}$ so pick $b_n = \frac{1}{n^2}$.

Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2 - n\sin(n)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2 - n\sin(n)} = 1$$

Since

$$\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } (p\text{-series}, p=2),$$

we conclude that

$$\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n^2 - n\sin(n)} \text{ also converges}$$

#215 Here $a_n = \frac{1}{n^{1+\frac{1}{n}}}$ so pick $b_n = \frac{1}{n}$. Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{1+1/n}}$$

$$= \lim_{n\to\infty} \frac{1}{n^{1/n}}$$

$$= \lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} e^{\ln(n^{-1/n})}$$

$$= \lim_{n\to\infty} e^{-\frac{1}{n}\ln(n)}$$

$$= e^{-\lim_{n\to\infty} \frac{\ln(n)}{n}} = e^0 = 1$$

But $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic... p-series w/ $p=1$)

So, $\sum a_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ also diverges.

#224 $\displaystyle\sum_{n=1}^{\infty} \frac{2^{pn}}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2^p}{3}\right)^n$

geometric series with $r = \frac{2^p}{3}$. We know it converges when $|r|<1$, i.e.

$$\left|\frac{2^p}{3}\right| < 1 \longrightarrow \frac{2^p}{3} < 1$$

$$\to 2^p < 3$$

$$\to p\ln(2) < \ln(3)$$

$$\to p < \frac{\ln(3)}{\ln(2)}$$

#234 here $a_n = \left(\ln(n)\right)^{-\ln(\ln(n))}$ so pick $b_n = \frac{1}{n}$.

Compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln(n))^{-\ln\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \underbrace{n(\ln(n))^{-\ln(\ln(n))}}_{\text{since } \log\log \ll \log\text{, this goes} \to \infty}$$

∞

$$= \infty$$

Since $\sum b_n = \sum \frac{1}{n}$ diverges, we conclude that

$$\sum a_n = \sum_{n=2}^{\infty} (\ln(n))^{-\ln\ln(n)} \quad \text{diverges}$$