Honors HW7 (due 3 April)

The Cantor set, $\mathcal{C}$ (named after the mathematician Georg Cantor), is constructed as follows: we start with the closed interval $[0,1]$ and remove the open interval $\left(\frac{1}{3},\frac{2}{3}\right)$ (the "middle third" of $[0,1]$), leaving us with $\left[0,\frac{1}{3}\right]$ and $\left[\frac{2}{3},1\right]$. Now remove the middle third of that set; four intervals remain. Remove middle thirds from each of the remaining 8 intervals, and so on. Continue this procedure infinitely, at each step removing the open middle thirds of the previous. The set that remains after this procedure continues to infinity is called $\mathcal{C}$ (sometimes referred to as "Cantor dust").

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1. Draw and label the endpoints of the first five steps of the construction of the Cantor set.
2. Write three numbers that remain in the Cantor set after the procedure continues to infinity.
3. Show that the total length of all the intervals that are removed is 1 (hint: geometric series).

The Cantor set has many interesting properties. It is "measure zero" in that it has no length and, relatedly, it is "totally disconnected" meaning it contains no intervals (if there were an interval, you would have removed its middle third at some point etc). It is an "uncountable" set, which means the number of elements in $\mathcal{C}$ is the same as the number of elements in the real numbers $\mathbb{R}$ (and in fact, this is a strictly larger infinity than the number of elements in $\{0,1,2,3,\ldots\}$ which is called "countable infinity"). Also it is a "perfect" set meaning that every point inside of $\mathcal{C}$ can be written as a limit of points in $\mathcal{C}$. It also contains some points that were not an endpoint at any step, for example the number $\frac{1}{4}\in\mathcal{C}$, but it is not an endpoint of any interval at any step!