MATH 1540 - EXAM 3 FALL 2019

SOLUTION

Friday, 22 November
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

Formulas

- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 1 - 2\sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \end{aligned}$$
- $\sin\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos(\alpha)}{2}}$
- $\cos\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1+\cos(\alpha)}{2}}$

1. (10 points) Prove the identity.

   (a) (5 points) $\cos(x) - \cos^3(x) = \cos(x)\sin^2(x)$

   Solution: Start with the left:

$$\cos(x) - \cos^3(x) \overset{\text{factor}}{=} \cos(x)\Big(1 - \cos^2(x)\Big) \underset{\cos^2(x)+\sin^2(x)=1}{\overset{1-\cos^2(x)=\sin^2(x)}{=}} \cos(x)\sin^2(x).$$

   (b) (5 points) $\dfrac{\sin(x+h) - \sin(x)}{h} = \sin(x)\left(\dfrac{\cos(h)-1}{h}\right) + \cos(x)\left(\dfrac{\sin(h)}{h}\right)$

   Solution: Using the sum identity for sine, start with the left and compute

$$\begin{aligned}\frac{\sin(x+h) - \sin(x)}{h} &= \frac{\Big(\sin(x)\cos(h) + \cos(x)\sin(h)\Big) - \sin(x)}{h} \\ &= \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h} \\ &= \sin(x)\left(\frac{\cos(h)-1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right).\end{aligned}$$

2. (14 points) Find the exact value.

   (a) (7 points) $\cos\left(\dfrac{5\pi}{12}\right)$

   Solution: Note that

$$\frac{\pi}{4} + \frac{\pi}{6} = \frac{3\pi}{12} + \frac{2\pi}{12} = \frac{5\pi}{12}.$$

   Therefore we may use the sum rule for cosine to compute

$$\begin{aligned}\cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

   (b) (7 points) $\cos\left(\dfrac{\pi}{8}\right)$

   Solution: Consider

$$\frac{\pi}{8} = \frac{\alpha}{2},$$

   implying that

$$\alpha = \frac{\pi}{4}.$$

   This means that we can use the half-angle identity:

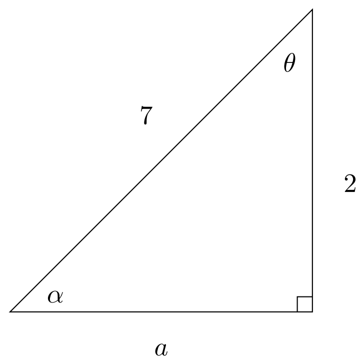
$$\begin{aligned}\cos\left(\frac{\pi}{8}\right) &= \cos\left(\frac{\frac{\pi}{4}}{2}\right) \\ &= +\sqrt{\frac{1 + \cos\left(\frac{\pi}{4}\right)}{2}} \\ &= \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} \\ &= \sqrt{\frac{2 + \sqrt{2}}{4}}.\end{aligned}$$

3. (6 points) Solve for θ in $[0, 2\pi)$: $2\sin(\theta) = 1$.
Solution: First divide by 2 to get

$$\sin(\theta) = \frac{1}{2}.$$

Now we look at the unit circle for all points whose second coordinate is $\frac{1}{2}$. That search reveals that $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$ are the solutions.

4. (14 points) Consider the following triangle:



Find

(a) (7 points) $\cos\left(\frac{\alpha}{2}\right)$
Solution: First find the missing leg a using the Pythagorean theorem:

$$a^2 + 2^2 = 7^2,$$

hence

$$a = \sqrt{49 - 4} = \sqrt{45}.$$

Now, using the half-angle identity,

$$\begin{aligned} \cos\left(\frac{\alpha}{2}\right) &= +\sqrt{\frac{1+\cos(\alpha)}{2}} \\ &= \sqrt{\frac{1+\frac{\sqrt{45}}{7}}{2}} \\ &= \sqrt{\frac{7+\sqrt{45}}{14}}. \end{aligned}$$

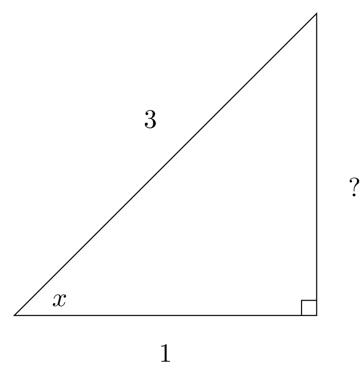
(b) (7 points) $\tan(2\theta)$

Solution: Using the double angle formulas, we get

$$\begin{aligned}
\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} \\
&= \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)-\sin^2(\theta)} \\
&= \frac{2\left(\frac{\sqrt{45}}{7}\right)\left(\frac{2}{7}\right)}{\left(\frac{2}{7}\right)^2-\left(\frac{\sqrt{45}}{7}\right)^2} \\
&= \frac{\frac{4\sqrt{45}}{49}}{\frac{4}{49}-\frac{45}{49}} \\
&= \left(\frac{4\sqrt{45}}{49}\right)\left(\frac{49}{-41}\right) \\
&= -\frac{4\sqrt{45}}{41}.
\end{aligned}$$

5. (14 points) If $\cos(x) = -\frac{1}{3}$ and x is in quadrant II, then find

   (a) (7 points) $\cos(2x)$
   Solution: First draw a triangle:



Using Pythagorean theorem, $1^2 + ?^2 = 3^2$, hence $? = \sqrt{9-1} = \sqrt{8}$. Therefore, $\sin(x) = \frac{\sqrt{8}}{3}$. Now we may compute

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \left(-\frac{1}{3}\right)^2 - \left(\frac{\sqrt{8}}{3}\right)^2 = \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}.$$

   (b) (7 points) $\sin\left(\frac{x}{2}\right)$
   Solution: Using the half-angle identity and the fact that if x is in QII, then $\frac{x}{2}$ is in QI,

$$\sin\left(\frac{x}{2}\right) = +\sqrt{\frac{1-\cos(x)}{2}} = \sqrt{\frac{1-\left(-\frac{1}{3}\right)}{2}} = \sqrt{\frac{\frac{4}{3}}{2}} = \sqrt{\frac{4}{6}} = \sqrt{\frac{2}{3}}.$$

6. (10 points) Consider the vectors $\vec{u} = \langle 1, 2\rangle$ and $\vec{v} = \langle -1, 3\rangle$.

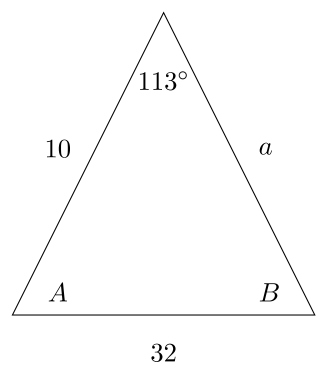
   (a) (5 points) Compute $\vec{u} + \vec{v}$.
   Solution: Calculate

$$\vec{u} + \vec{v} = \langle 1, 2\rangle + \langle -1, 3\rangle = \langle 1 + (-1), 2 + 3\rangle = \langle 0, 5\rangle.$$

(b) (5 points) Compute $3\vec{u} - 2\vec{v}$.
Solution: Calculate

$$3\vec{u} - 2\vec{v} \quad = 3\langle 1, 2\rangle - 2\langle -1, 3\rangle = \langle 3, 6\rangle + \langle 2, -6\rangle = \langle 5, 0\rangle.$$

7. (16 points) Solve the triangle. Express your answers with two decimal places.



Solution: **Find B**
By law of sines,

$$\frac{\sin(B)}{10} = \frac{\sin(113^\circ)}{32},$$

so multiplying by 10 yields

$$\sin(B) = \frac{10\sin(113^\circ)}{32}.$$

Taking inverse sine of both sides yields

$$B = \sin^{-1}\left(\frac{10\sin(113^\circ)}{32}\right) \approx 16.72^\circ.$$

(note: we need to check for a second solution. The second possible value of B would be

$$B_2 = 180^\circ - 16.72^\circ = 163.28^\circ,$$

but this value is **too large**, so we do not have a second solution to this triangle)

Find A
Using the sum of angles of a triangle being 180°, we see

$$113^\circ + 16.72^\circ + A = 180^\circ,$$

so

$$A = 180^\circ - 113^\circ - 16.72^\circ = 50.28^\circ.$$

Find a
Using law of sines,

$$\frac{\sin(50.28^\circ)}{a} = \frac{\sin(113^\circ)}{32}.$$

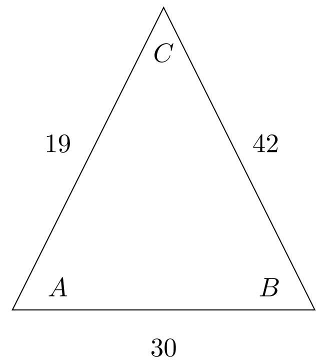
Therefore multiply by a and by 32 to get

$$32\sin(50.28^\circ) = a\sin(113^\circ),$$

and divide by the number $\sin(113^\circ)$ to arrive at

$$a = \frac{32\sin(50.28^\circ)}{\sin(113^\circ)} \approx 26.739.$$

8. (16 points) Solve the triangle. Express your answers with two decimal places.



Solution: **Find A**

Using law of cosines,

$$42^2 = 19^2 + 30^2 - 2(19)(30)\cos(A),$$

and algebra yields

$$\cos(A) = \frac{42^2 - 19^2 - 30^2}{-2(19)(30)},$$

so taking the inverse cosine gives us

$$A = \cos^{-1}\left(\frac{42^2 - 19^2 - 30^2}{-2(19)(30)}\right) \approx 116.2°.$$

Find B

Using law of cosines,

$$19^2 = 30^2 + 42^2 - 2(30)(42)\cos(B),$$

and hence

$$B = \cos^{-1}\left(\frac{19^2 - 30^2 - 42^2}{-2(30)(42)}\right) \approx 23.95°.$$

Find C

Using the sum of the angles of a triangle being 180°, we get

$$116.2° + 23.95° + C = 180°,$$

and so

$$C = 180° - 116.2° - 23.95° = 39.85°.$$