Series

FROM CALC 2} the idea: write a function f(x) as an "infinite polynomial" ~ given a fixed $c \in \mathbb{C}$,

$$f(z) = a_0 + a_1(z-c) + a_2(z-c)^2 + a_3(z-c)^3 + \dots$$
$$= \sum_{k=0}^{\infty} a_k (z-c)^k$$

The question: how to find numbers $a_0, a_1, a_2, \dots$ that makes this equality true?

TRICK: plug in $x=c$ to get

$$f(c) = a_0 + \underbrace{a_1(c-c)}_{=0} + \underbrace{a_2(c-c)^2}_{=0} + \dots$$

$$\boxed{f(c) = a_0}$$

Now take derivative:

$$f'(z) = 0 + a_1 + 2a_2(z-c)^1 + 3a_3(z-c)^2 + 4a_4(z-c)^3 + \dots$$

$$\boxed{f'(c) = a_1} + 0 + 0 \dots$$

In general can show by induction that

$$\boxed{f^{(n)}(c) = n!\,a_n}$$

and so we get Taylor series centered at c:

$$\boxed{f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(z-c)^k}$$

and the outlined process works whenever $f^{(n)}(c)$ exists for any $n \in \{0,1,2,\dots\}$

Ex: find Taylor series of e^z . . .

a) centered at $c=0$

b) centered at $c=1$

Convergence

What does it mean for a series $z_0+z_1+\cdots$ to exist?

Def: The series $\sum_{n=0}^{\infty} z_n = z_0+z_1+\cdots$ converges to the sum S provided that the sequence $S_n \to S$ where S_n is the partial sum $S_n = \sum_{k=0}^{n} z_k = z_0+z_1+\dots+z_n$.

FACT: (geometric series) Consider $\sum_{k=0}^{\infty} z^k$

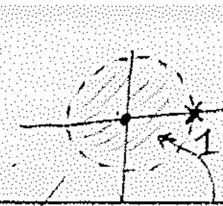
① Show by induction that

$$\sum_{k=0}^{n} z^k = 1+z+\dots+z^n = \frac{1-z^{n+1}}{1-z} \quad (z \neq 1)$$

② Consider the "tails" of the series:

$$\text{tail}_N(z) = \left(\sum_{k=0}^{\infty} z^k\right) - \left(\sum_{k=N}^{\infty} z^k\right) = \sum_{k=0}^{N-1} z^k \overset{①}{=} \frac{1-z^N}{1-z}$$

③ $$|\text{tail}_N(z)| \le \frac{1-|z|^N}{|1-z|} \longrightarrow 0$$
iff $|z|<1$



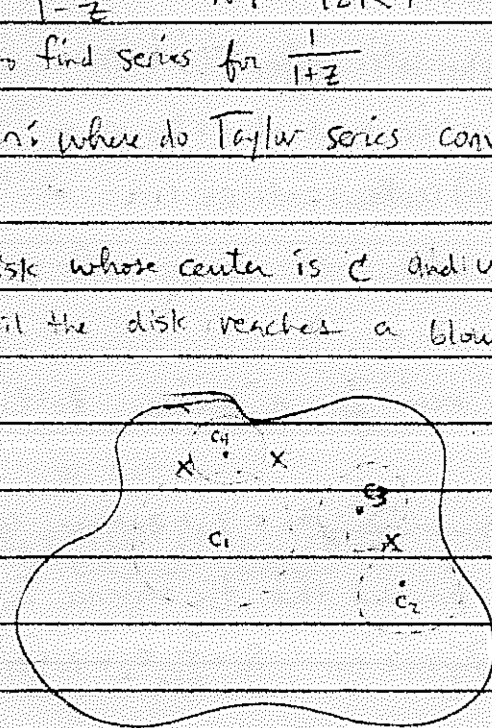
Therefore we can conclude "region of convergence"

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad \text{iff} \quad |z|<1$$

Ex: Use above to find series for $\frac{1}{1+z}$

Natural question: where do Taylor series converge?

Answer: in a disk whose center is c and whose radius extends until the disk reaches a blow-up point



Ex: Find Taylor series centered at 0 for a) $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

b) $\sinh(z)$ (recall: $\sinh(z) = -i\sin(iz$

c) $\frac{1}{1+z^2}$ (use geo)

Ex: Find "a" series for $f(z) = \frac{1+2z^2}{z^3+z^5}$ ⟵ note: can't use Taylor s with $c=0$ b/c not analytic there

Soln: $f(z) = \frac{1}{z^3}\left(\frac{1+2z^2}{1+z^2}\right) = \frac{1}{z^3}\left(\frac{2(1+z^2)-1}{1+z^2}\right) = \left(2 - \frac{1}{1+z^2}\right)\frac{1}{z^3}$

make appear

$$= \frac{2}{z^3} - \frac{1}{z^3}\cdot\frac{1}{1+z^2}$$

use earlier
$= \sim$

Laurent Series

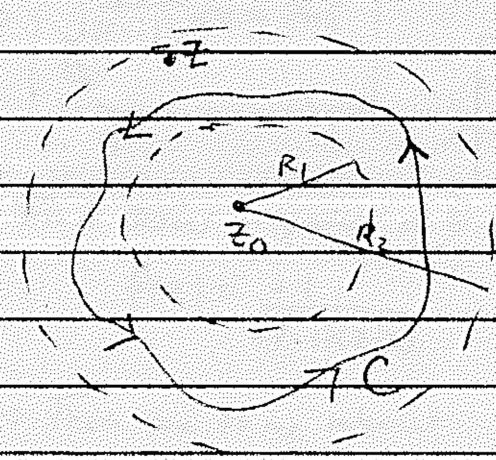
We need these to write series of functions centered at a blow-up point.

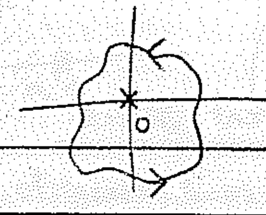
Laurent Theorem: Suppose f is $\mathbb{C}$-diff'bl in an annulus $R_1 < |z-z_0| < R_2$ centered at z_0, and let C denote any $\oplus$ oriented simple closed contour lying in the annulus. Then, at each point z in the annulus,

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$$

where

$$\begin{cases} c_k = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f(z)}{(z-z_0)^{k+1}}\, dz \\ k \in \mathbb{Z} \end{cases}$$





Ex: Find Laurent series for $e^{\frac{1}{z}}$

Soln: Blow-up point is at $z=0$.

Consider

$$e^{z}=\sum_{k=0}^{\infty}\frac{z^k}{k!},$$ ← Taylor series centered at $z=0$

so

$$e^{\frac{1}{z}}=\sum_{k=0}^{\infty}\frac{1}{z^k k!}=1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!z^3}+\cdots$$

By the theorem we have

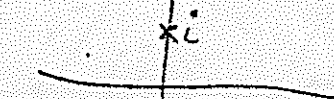
$$c_n=\begin{cases}0, & n=1,2,3,\dots\\ \frac{1}{|n|!}, & n=0,-1,-2,\dots\end{cases}$$

and,

$$\underset{\substack{\uparrow\\ \text{from formula above}}}{1=}c_{-1}\underset{\substack{\uparrow\\ \text{theorem}}}{=}\frac{1}{2\pi i}\int_C\frac{e^{\frac{1}{z}}}{(z-z_0)^{-1+1}}dz=\frac{1}{2\pi i}\int_C e^{1/z}dz,$$

hence

$$\boxed{\int_C e^{1/z}dz=2\pi i}$$



Ex: Consider $f(z) = \frac{1}{(z-i)^2}$ Find Laurent series centered at i.

This f has blow-up point at $z=i$. Also it is already written as a Laurent series!! So

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-i)^k \rightsquigarrow \text{only } k=-2 \text{ term has } c_k \neq 0 : c_{-2} = 1$$

$$c_k = \begin{cases} 0, & k \in \mathbb{Z} \setminus \{-2\} \\ 1, & k=-2 \end{cases}$$

So,

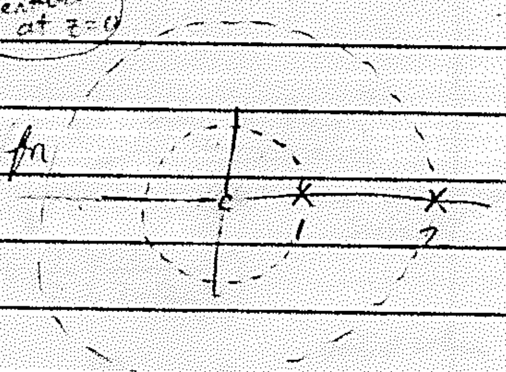
$$0 = c_{-1} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-i)^{-1+1}} dz = \frac{1}{2\pi i} \int_C \frac{1}{(z-i)^2} dz$$

$$\Rightarrow \boxed{\int_C \frac{1}{(z-i)^2} dz = 2\pi i}$$

centered at $z=0$

Ex: Find series in powers of z for

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



Soln: There are three relevant domains:

(A) $|z| < 1$

(B) $1 < |z| < 2$

(C) $2 < |z| < \infty$

Each will have a different series representation.

First use partial fractions decomp on f:

$$f(z) = \frac{-1}{(z-1)(z-2)} = \overbrace{\frac{A}{z-1} + \frac{B}{z-2}}^{\text{"guess"}}$$

Then

$$-1 = (A)(z-2) + (B)(z-1)$$

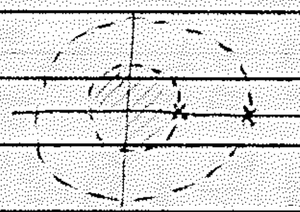
$$-1 = (A+B)z + (-2A-B) \Longrightarrow \begin{cases} A+B=0 \rightarrow A=-B \\ -2A-B=-1 \rightarrow 2B-B=-1 \end{cases}$$

$$\boxed{B=-1} \qquad \boxed{A=1}$$

Thus

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2-z}$$

For Ⓐ: $|z|<1$



From geo series,

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

why do this?
So we can use geo series!!

We see

$$\frac{1}{z-2} = \left(\frac{1}{2-z}\right)\left(\frac{1/2}{1/2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{1-(z/2)}\right)$$

$$\overset{\text{geo series}}{=} \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k$$

valid when $\left|\frac{z}{2}\right|<1$, i.e. $|z|<2$, which is ok!!

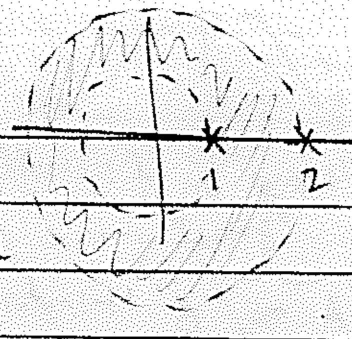
$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k}$$

Thus for Ⓐ,

$$f(z) = \frac{1}{1-z} + \frac{1}{2-z} = \left(\sum_{k=0}^{\infty} z^k\right) + \left(\frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k}\right).$$

For (B)

$1<|z|<2$



Can't use "$\frac{1}{1-z} = \sum z^k$" anymore

So rewrite

$$f(z) = \frac{1}{1-z} + \frac{1}{2-z} = \frac{1}{z}\frac{1}{\frac{1}{z}-1} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) + \frac{1}{2}\left(\frac{1}{1-z/2}\right)$$

now notice: Since $1<|z|<2$

$1 > \left|\frac{1}{z}\right| > \frac{1}{2}$ and $\left|\frac{z}{2}\right| < 1$

can use geo series

can use geo series

So we get

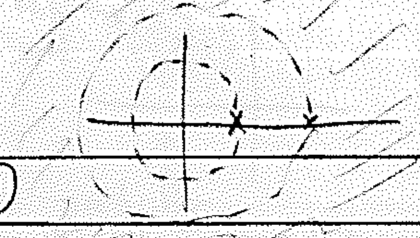
$$f(z) = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k$$

$$= -\sum_{k=0}^{\infty}\frac{1}{z^{k+1}} + \sum_{k=0}^{\infty}\frac{z^k}{2^k}$$

reindex

$$= \left(\sum_{k=1}^{\infty}\frac{-1}{z^k}\right) + \left(\sum_{k=0}^{\infty}\frac{z^k}{2^k}\right)$$

$\longleftarrow$ notice this is a Laurent series — compare to the thm



For (c)

$|z|>2$ — if $|z|>2$, then

$$f(z) = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)} \qquad \left|\frac{1}{z}\right| < \frac{1}{2} \checkmark \quad \left|\frac{2}{z}\right| < 1 \checkmark$$

ok for geo

$$= \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k - \frac{1}{z}\sum_{k=0}\left(\frac{2}{z}\right)^k$$

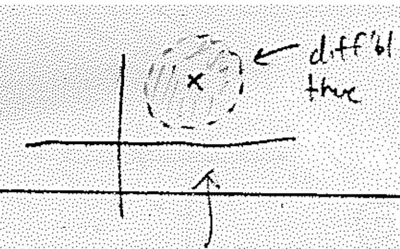
$$= \sum_{k=0}^{\infty}\frac{1}{z^{k+1}} - \sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}}$$

$$\overset{\text{reindex}}{=} \sum_{k=1}^{\infty}\left(\frac{1}{z^k} - \frac{2^{k-1}}{z^k}\right) = \sum_{k=1}^{\infty}\left(\frac{1-2^{k-1}}{z^k}\right)$$

So to summarize:

$$\frac{-1}{(z-1)(z-2)} = f(z) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} z^k + \frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k} & ; |z|<1 \\ \displaystyle\sum_{k=1}^{\infty}\frac{-1}{z^k} + \sum_{k=0}^{\infty}\frac{z^k}{2^k} & ; 1<|z|<2 \\ \displaystyle\sum_{k=1}^{\infty}\frac{1-2^{k-1}}{z^k} & ; |z|>2 \end{cases}$$

Residues and poles



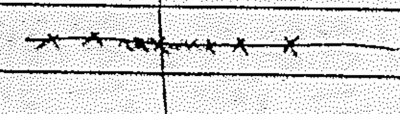
<u>Def</u>: A blow-up point z_0 of f is called <u>isolated</u> if there is a deleted neighborhood $0 < |z - z_0| < \epsilon$ such that f is $\mathbb{C}$-diff'bl there.

<u>Ex</u>: $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has 3 isolated blow-up points at $z = 0, \pm i$

Ex: $f(z) = \text{Log}(z)$ has a blow-up pt at $z=0$ BUT it is not isolated b/c of branch cut

Ex: $f(z) = \dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$ has blow-up points at $z=0$ and $z = \frac{1}{n}$ for $n = \pm 1, \pm 2, \ldots$

but the one at $z=0$ is not isolated



Recall Laurent theorem: if z_0 is isolated blow-up point, then for z obeying $0 < |z| < R$ (for some R), and C lying in there,

$$\begin{cases} f(z) = \displaystyle\sum_{k=-\infty}^{\infty} c_k (z-z_0)^k \\[2ex] c_k = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z)\,dz}{(z-z_0)^{k+1}} & k < 0 \\[2ex] \dfrac{1}{2\pi i} \displaystyle\int_C f(z)(z-z_0)^{k+1}; & k \geq 0 \end{cases} \end{cases}$$

So in particular,

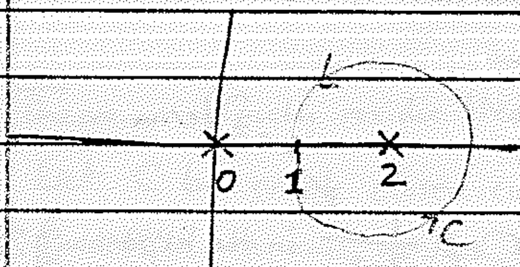
$$c_{-1} = \frac{1}{2\pi i} \int_C f(z)\, dz$$

coeff of $\frac{1}{z-z_0}$ in Laurent series

We call the value $2\pi i c_{-1}$ the residue of f at the isolated singular point z_0, and we write

$$\operatorname*{Res}_{z=z_0} f(z) = 2\pi i c_{-1} = \int_C f(z)\, dz$$

Ex: Consider $\int_C \frac{1}{z(z-2)^4}\, dz$ where C is $|z-2|=1$



Strategy: find Laurent series centered at 2 and then use Laurent theorem to compute the $\int$

So, need to write

$$\frac{1}{z(z-2)^4} = \sum_{k=-\infty}^{\infty} c_k (z-2)^k$$

Since $\frac{1}{(z-2)^4}$ is already ok, we focus on

$$\frac{1}{z} = \frac{1}{z+2-2} = \frac{1}{2+(z-2)} = \frac{1/2}{1+\left(\frac{z-2}{2}\right)} = \frac{1/2}{1-\left(-\frac{z-2}{2}\right)}$$

Now as long as $\left|-\frac{z-2}{2}\right| < 1 \longleftrightarrow |z-2| < 2 \longleftrightarrow$

we know that

$$\frac{1}{z} = \frac{1}{2}\,\frac{1}{1-\left(-\frac{z-2}{2}\right)} = \frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{(z-2)}{2}\right)^k$$

and so we have

$$\frac{1}{z}\cdot\frac{1}{(z-2)^4} = \frac{1}{(z-2)^4}\cdot\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k (z-2)^k}{2^k}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k (z-2)^{k-4}}{2^{k+1}}$$

Thus,

$$\operatorname*{Res}_{z=2} \frac{1}{z(z-2)^4} = \left(\text{coeff of } \underbrace{\frac{1}{z-2}}_{k=3 \text{ term}} \text{ in the expansion}\right) 2\pi i$$

$$= 2\pi i\left(\frac{(-1)^3}{2^4}\right) = \frac{-\pi i}{8}$$

and so $\displaystyle\int_C \frac{1}{z(z-2)^4}\,dz = \frac{-\pi i}{8}$

Ex: Earlier we saw $\int_C e^{1/z}\,dz = 2\pi i$ where C was unit circle. Now consider

$$\int_C e^{\frac{1}{z^2}}\,dz$$

Soln: From

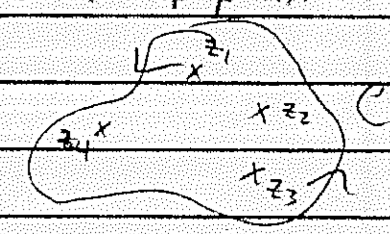
$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \text{ we get}$$

$$e^{1/z^2} = \sum_{k=0}^{\infty} \frac{1}{z^{2k}\,k!} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \cdots$$

and so

$$\int_C e^{1/z^2}\,dz = 2\pi i \underbrace{\operatorname*{Res}_{z=0} f(z)}_{\text{coeff of } \frac{1}{z}} = 0$$

Now we can handle more than one blow-up point in the interior of a contour.



Theorem (Cauchy residue theorem):
Let C be simple closed curve $\oplus$ oriented. If f is $\mathbb{C}$-diff'bl on and inside C except for the finitely many points $z_1, \ldots, z_n$ inside C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)}$$