Recall if C is a contour then $-C$ is same contour but in opposite direction and

$$\int_{-C} f\,dz = -\int_C f\,dz$$

Recall fundamental theorem of calculus from Calc 1:

$$\int_a^b f'(t)\,dt = f(t)\Big|_a^b = f(b) - f(a)$$

— i.e. the integral of f' only depends on what f does at a and at b

(note: Calc 3 ~ FTLI, green's theorem and stokes theorem generalize this)

Def: An antiderivative of a function $f: D \subseteq \mathbb{C} \to \mathbb{C}$ is a function $F: D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in \mathbb{C}$

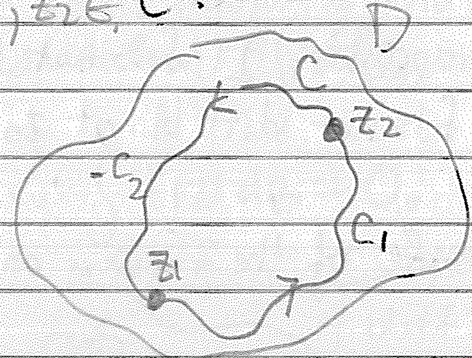
Theorem: Suppose $f: D \to \mathbb{C}$ is continuous. The following are equivalent:

(i) f has antiderivative F in D

(ii) if $z_1, z_2 \in D$ and C_1 and C_2 are ANY contours from z_1 to z_2 lying in D, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

(iii) the integral around any closed contour lying entirely inside D is equal to zero

Proof that (ii) $\to$ (iii): Assume (ii) holds — we will prove (iii).
Consider any closed contour C in D and pick any $z_1, z_2 \in C$:



Call portion from z_1 to z_2 C_1 and portion from z_2 to z_1 $-C_2$.

By (ii),

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz,$$

so

$$(\star) \quad \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$

Now calculate

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

$$\overset{(\star)}{=} 0,$$

as was to be shown. ■

Theorem: If $f: D \to \mathbb{C}$ is diff'bl at $z_0 \in D$, then f is continuous at z_0.

Corollary: (contrapositive) If f is not continuous at z_0, then f is not diff'bl at z_0.

Illuminating example of the Theorem on p. 66:

Let C = the unit circle. We have seen

$$\int_C z^n \, dz = 0 \quad \text{and} \quad \int_C \frac{1}{z} \, dz = 2\pi i \neq 0$$
$$n \in \mathbb{Z} \setminus \{-1\}$$

Why is this not a contradiction to the theorem?

< discuss as a class >

Notice: the theorem on p. 66 does not tell us a condition for f that guarantee that (i), (ii), and (iii) hold, only that they all must be simultaneously true or simultaneously false

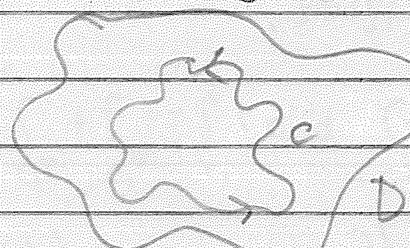
We now write such a theorem

aka "Cauchy integral theorem"

Theorem: (Cauchy-Goursat Theorem)

If a function f is complex-differentiable on all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$



diff'bl on boundary of D and interior

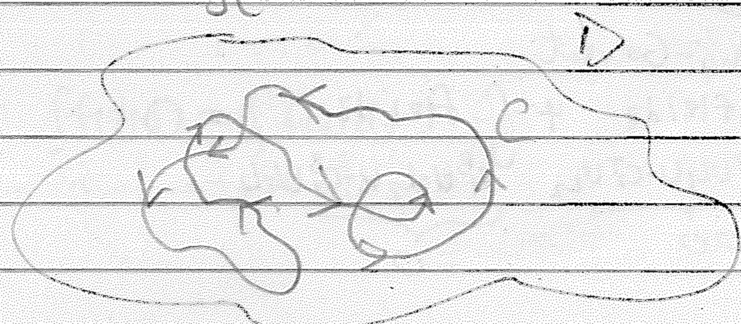
$\downarrow$

$\int_C f(z)\,dz = 0$

Extension of Cauchy-Goursat:

Theorem: If f complex-diff'bl throughout a simply connected domain D, then for every closed contour C lying in D,

$$\int_C f(z)\,dz = 0.$$

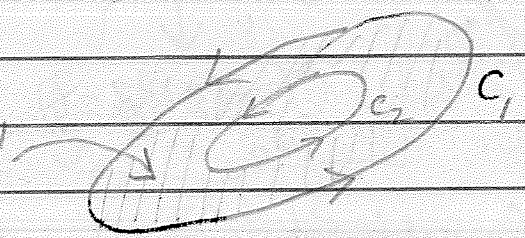


Deformation of paths:

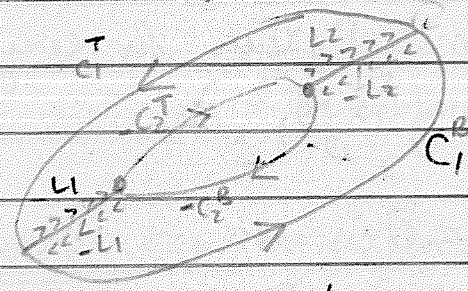
<u>Theorem</u>: Let C_1 and C_2 be positively oriented closed contours where C_2 lies entirely inside C_1. If f is $\mathbb{C}$-diff'bl on those contours and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

f diff'bl here



$\Downarrow$ introduce some segments, change dir of C_2



By Cauchy-Goursat

$$(\ast)\quad \underbrace{\int_{C_1^T \cup L_1 \cup -C_2^T \cup L_2} f(z)\,dz}_{=0} + \underbrace{\int_{C_1^B \cup -L_2 \cup -C_2^B \cup -L_1} f(z)\,dz}_{=0} = 0$$

But,

$$\int_{C_1^T \cup L_1 \cup -C_2^T \cup L_2} f(z)\,dz = \left[\int_{C_1^T} + \int_{L_1} + \int_{-C_2^T} + \int_{L_2}\right] f(z)\,dz$$

and

$$\int_{C_1^B \cup -L_2 \cup -C_2^B \cup -L_1} f(z)\,dz = \left[\int_{C_1^B} + \int_{-L_2} + \int_{-C_2^B} + \int_{-L_1}\right] f(z)\,dz$$

Since

$$\int_{C_1^T} + \int_{C_1^B} = \int_{C_1} \text{ and } \int_{-C_2^T} + \int_{-C_2^B} = \int_{-C_2}, \text{ we have}$$

by $(*)$,

negatives → add to 0

$$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz + \underbrace{\int_{L_1} f(z)\,dz + \int_{-L_1} f(z)\,dz}_{\text{zero}} + \int_{L_2} f(z)\,dz + \int_{-L_2} f(z)\,dz = 0,$$

hence

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

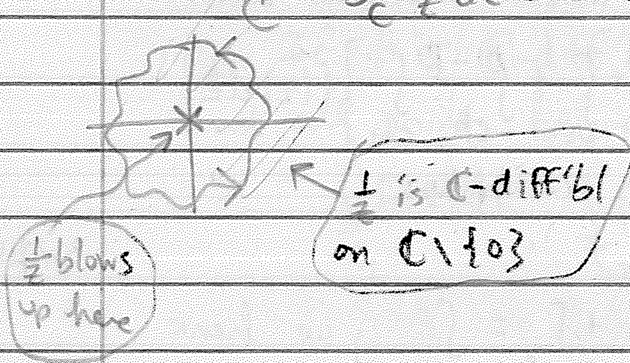
$\Rightarrow$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Recall: We showed if C was unit circle, then $\int_C \frac{1}{z}\,dz = 2\pi i$

Ex: By deformation of paths, if C is any closed contour surrounding 0, then oriented $\oplus$

$$\int_C \frac{1}{z}\,dz = 2\pi i$$



Examples:

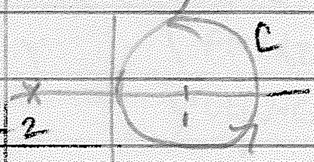
If C is any closed contour and $f(z)$ is any polynomial, then

$$\int_C f(z)\,dz = 0,$$

since all polynomials are diff'bl everywhere!!

Ex: Let C be the circle $|z-1| = 1$ oriented $\oplus$

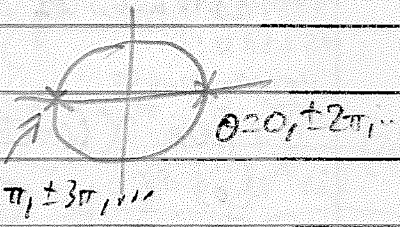
Then $\int_C \frac{1}{z+2}\,dz = 0$



$\leftarrow$ $\frac{1}{z+2}$ diff'bl everywhere except at $z=-2$, which is far away from C

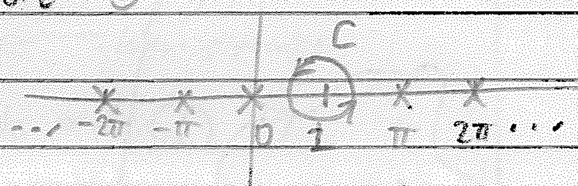
Fact: $\sin(z)=0 \longleftrightarrow \dfrac{e^{iz}-e^{-iz}}{2i}=0$

$$\longleftrightarrow e^{iz}=e^{-iz}$$
$$\longleftrightarrow e^{2zi}=1$$
$$\longleftrightarrow z=n\pi$$
$$n\in\mathbb{Z}$$



Ex: If C is circle $|z-1|=\frac{1}{100}$, oriented ⊕, then

$$\int_C \frac{5z^2+10z-1}{\sin(z)}\,dz=0$$

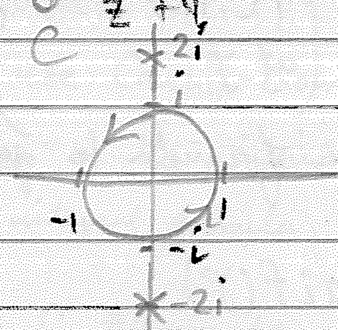


Ex: If C is unit circle oriented ⊕, then

$$\int_C \frac{1}{z^2+4}\,dz=0$$

$z^2+4=0$

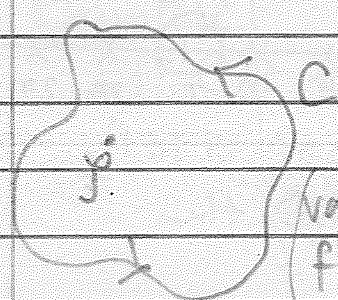
$z=\pm 2i$



Now we arrive at one of the most important theorems of complex calculus.

Theorem (Cauchy integral formula)
Let f be $\mathbb{C}$-diff'bl everywhere on and inside a simple closed contour C oriented ⊕. If ζ is any point in the interior of C, then

$$f(\zeta) = \frac{1}{2\pi i} \underbrace{\int_C \frac{f(z)}{z-\zeta}\, dz}$$



$$\begin{pmatrix}\text{value of}\\ f \text{ at } \zeta\end{pmatrix} = \frac{1}{2\pi i}\begin{pmatrix}\text{integral of a}\\ \text{certain function}\\ \text{on } C \text{ that blows up}\\ \text{at } \zeta\end{pmatrix}$$

Note we can rearrange the formula to get

$$\int_C \frac{f(z)}{z-\zeta}\, dz = 2\pi i f(\zeta)$$

can help us integrate faster!!

Ex: Let C be circle $|z|=2$ oriented $\oplus$.

Calculate

$$\int_C \frac{z}{(9-z^2)(z+i)}\,dz$$

old way $\begin{cases} z(t) = 2e^{it} \\ 0 \le t \le 2\pi \end{cases}$

$\Rightarrow$ must integrate

$$\int_0^{2\pi} \frac{2e^{it}}{(9-e^{2it})(e^{it}+i)}(2ie^{it})\,dt$$

good luck with that!!

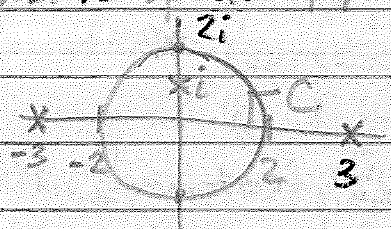
Soln: Find where integrand blows up:

$$(9-z^2)(z+i)=0$$

$$9-z^2=0 \qquad z+i=0$$

$$z=\pm 3 \qquad z=-i \longleftarrow \text{blow-up points}$$

Draw picture of blow up points and C:



Cannot use Cauchy-Goursat here! The function is not analytic on the interior of C! However, by the Cauchy integral formula,

$$f(z) = \frac{z}{9-z^2}$$

$$\int_C \frac{z}{(9-z^2)(z+i)}\,dz = \int_C \frac{\left(\frac{z}{9-z^2}\right)}{z-(-i)}\,dz = 2\pi i\, f(-i)$$

$$= 2\pi i\left(\frac{-i}{9-(-i)^2}\right)$$

$$= \frac{2\pi}{10} = \frac{\pi}{5}$$

z_0

WOW!!

Ex: Let C be unit circle oriented (+).

Compute
$$\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}\,dz$$

Compute
$$\int_C \frac{\cos(z)}{z(z^2+8)}\,dz$$

Compute
$$\int_C \frac{z}{2z+1}\,dz$$

Derivatives of analytic functions

Consider the following from calc 1:
$$\lim_{x\to a} f(x)=L \quad \text{iff} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

① The function $f(x)=|x|=\begin{cases}-x; & x<0\\ x; & x\geq 0\end{cases}$ is not diff'bl at $x=0$:

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h-0}{h} = \lim_{h\to 0^-} -1 = -1,$$ NOT equal!

but
$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h-0}{h} = \lim_{h\to 0} 1 = 1$$

(2) The function $g(x) = \begin{cases} -\frac{x^2}{2} & ; x<0 \\ \frac{x^2}{2} & ; x \geq 0 \end{cases}$ IS diff'bl at $x=0$.

$$\lim_{h\to 0^-} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^-} \left(-\frac{h^2}{2}\right)\cdot\frac{1}{h} = 0$$
$$\lim_{h\to 0^+} \frac{g(h)-g(0)}{h} = \lim_{h\to 0^+} \left(\frac{h^2}{2}\right)\cdot\frac{1}{h} = 0$$
equal!

But $g'(x) = \begin{cases} -\frac{2x}{2} = -x & ; x<0 \\ \frac{2x}{2} = x & , x>0 \end{cases} = |x|,$

so we know that g is <u>not</u> twice diff'bl at $x=0$.

The situation is much different in complex calculus. By Cauchy Integral formula, if $f: D \to \mathbb{C}$ is diff'bl at $\zeta \in D$, then for certain contours C,

$$f(\zeta) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}\,dz$$

Taking the derivative w.r.t. ζ gives us

(technically ~ should be justified)

$$f'(\zeta) = \frac{d}{d\zeta}f(\zeta) = \frac{1}{2\pi i}\int_C \frac{d}{d\zeta}\frac{f(z)}{z-\zeta}\,dz$$

$$= \frac{1}{2\pi i}\int_C f(z)\frac{d}{d\zeta}\left(\frac{1}{z-\zeta}\right)dz$$

$$\frac{d}{d\zeta}(z-\zeta)^{-1} = -(z-\zeta)^{-2}(-1) = \frac{1}{(z-\zeta)^2}$$

chain rule

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\zeta)^2}\,dz$$

Similarly,

$$f''(\zeta) = \frac{1}{2\pi i}\int_C f(z)\,\frac{d}{d\zeta}\frac{1}{(z-\zeta)^2}\,dz$$

$$= \frac{2}{2\pi i}\int_C \frac{f(z)}{(z-\zeta)^3}\,dz$$

And by induction, can show

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-\zeta)^{n+1}}\,dz$$

The consequence is...

Theorem: If a function is $\mathbb{C}$-diff'bl at a point, then it is infinitely $\mathbb{C}$-diff'bl at that point.

Combining the formula above with ML-inequality gives us

Lemma: If f is $\mathbb{C}$-diff'bl inside and on a $\oplus$ oriented circle C_R centered at z_0 with radius R and M_R denotes the maximum value of $|f(z)|$ on C_R, then

$$|f^{(n)}(z_0)| \leq \frac{n!\,M_R}{R^n}$$

means $|f(z)|<M$ for some M for all $z\in\mathbb{C}$

Theorem (Liouville's theorem) If f is $\mathbb{C}$-diff'bl at ALL points of $\mathbb{C}$ and also bounded, then f is constant.

Proof: By the lemma, for any $z_0 \in \mathbb{C}$ and $R>0$,

$$|f'(z_0)| \le \frac{M_R}{R}$$

Since f is bounded, there is a constant M (not dependent on R) such that for all $z\in\mathbb{C}$, $|f(z)|\le M$. Since $M \le M_R$ (why?), we see

$$|f'(z_0)| \le \frac{M}{R}$$

Since M is independent of R, taking the limit as $R\to\infty$ of both sides shows

$$|f'(z_0)| \le 0$$

Since the modulus is ≥ 0 always, we conclude

$$0 \le |f'(z_0)| \le 0$$

and hence $f'(z_0)=0$. Even more, we assumed z_0 was arbitrary, so $f'(z)=0$ for all $z\in\mathbb{C}$.

Therefore $f:\mathbb{C}\to\mathbb{C}$ is a constant function. ∎

Theorem: (Fundamental theorem of algebra)
Any polynomial
$$P(z) = a_0 + a_1 z + \dots + a_n z^n \quad (a_n \neq 0)$$
of degree $n \geq 1$ has at least one root.

Proof: Suppose $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then
$$f(z) = \frac{1}{P(z)}$$
is $\mathbb{C}$-diff'bl everywhere.

Claim: f is bounded

Proof of claim:
$$f(z) = \frac{1}{a_0 + a_1 z + \dots + a_n z^n}$$
$$= \underbrace{\left(\frac{1}{z^n}\right)}_{\left(\substack{\to 0 \\ \text{as} \\ z \to \infty}\right)} \underbrace{\left(\frac{1}{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + a_n}\right)}_{\left(\to \frac{1}{a_n} \text{ as } z \to \infty\right)} \xrightarrow{z \to \infty} 0$$

Therefore for sufficiently large z, say those outside of a circle $|z| = R$, $|f(z)| \leq 1$.

Inside the circle $|z| = R$, f is continuous and hence also bounded there.

Since f $\mathbb{C}$-diff'bl and bdd in $\mathbb{C}$, Liouville's theorem says that f is constant, a contradiction. $\blacksquare$