Recall: Δ-inequality : $|z_1+z_2| \le |z_1|+|z_2|$

another Δ-inequality : $|z_1 \pm z_2| \ge \big||z_1|-|z_2|\big|$

and $a \ge b \Rightarrow \frac{1}{a} \le \frac{1}{b}$

Bounds of contour integrals

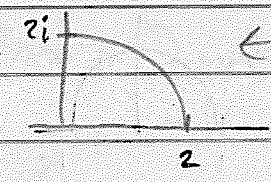
A common calculation:

$$(\star) \quad \left|\int_C f(z)\,dz\right| \underset{\substack{\uparrow \\ C:\begin{cases} z(t) \\ a\le t\le b\end{cases}}}{=} \left|\int_a^b f(z(t))\,z'(t)\,dt\right| \underset{\substack{\uparrow \\ \Delta\text{-inequality} \\ (\int \text{ is a sum...})}}{\le} \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

Now if f satisfies $|f(z)| \le M$ for any $z \in C$, then $(\star)$ becomes

$$\left|\int_C f(z)\,dz\right| \le M \underbrace{\int_a^b |z'(t)|\,dt}_{\text{arc length of } C} = ML$$

Example: Estimate $\left|\int_C \frac{z+4}{z^3-1}\,dz\right|$ where C is the part of the circle $|z|=2$ in quadrant .

Soln: Here, C is 

$\leftarrow$ length $\boxed{L=\pi}$ ($\frac{1}{4}$ of 4π - the circumference)

Now, how to bound $\left|\dfrac{z+4}{z^3-1}\right|$?

Numerator: $|z+4| \underset{\Delta\text{-ineq}}{\leq} |z|+4 = 2+4 = 6$

$|z|=2$ on C

denominator: $|z^3-1| \geq \big|\,|z|^3 - 1\,\big| = |2^3-1| = 7$

$|z|=2$ on C

$|z^3|=|z|^3$

Therefore
$$\frac{1}{|z^3-1|} \leq \frac{1}{7}$$

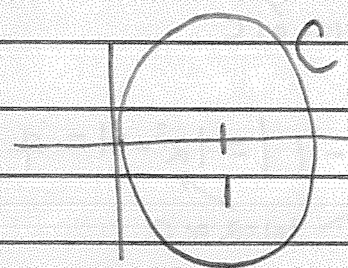
Therefore
$$\left|\frac{z+4}{z^3-1}\right| \leq \boxed{\frac{6}{7} =: M}$$

Thus by ML-inequality,

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \leq ML = \frac{6}{7}\pi$$

Ex: Estimate $\left|\int_C \frac{2z+1}{(z-1)^4-17}\,dz\right|$ where C is the circle $|z-1|=1$.

Soln:



~ length is 2π (circumf) ← L

Bound $\left|\frac{2z+1}{(z-1)^4-17}\right|$

numerator: $|2z+1| = |2(z-1+1)+1|$

$= |2(z-1)+3|$

$\leq 2|z-1|+3$

circle is $|z-1|=1$

$= 2+3 = 5$

denom: $|(z-1)^4-17| \geq \left|\,|z-1|^4-17\right| = |1^4-17| = |-16| = 16$

Therefore, $\left|\frac{2z+1}{(z-1)^4-17}\right| \leq \frac{5}{16}$ ← M. Hence by ML-ineq,

$$\left|\int_C \frac{2z+1}{(z-1)^4-17}\,dz\right| \leq ML = \frac{5}{16}\cdot 2\pi = \frac{5\pi}{8}$$