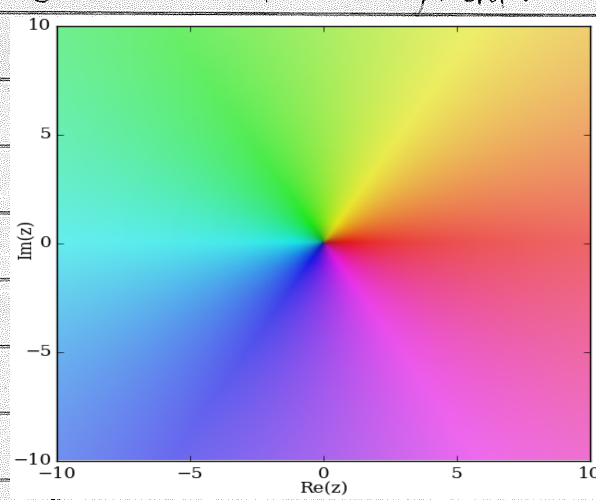
Domain Colorings

To plot a function $\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = \sim \end{cases}$, you form all of the ordered pairs $(x, f(x))$ and plot them. This is possible because one-dimensional input and one-dimensional output $\Rightarrow$ two dimensional graph

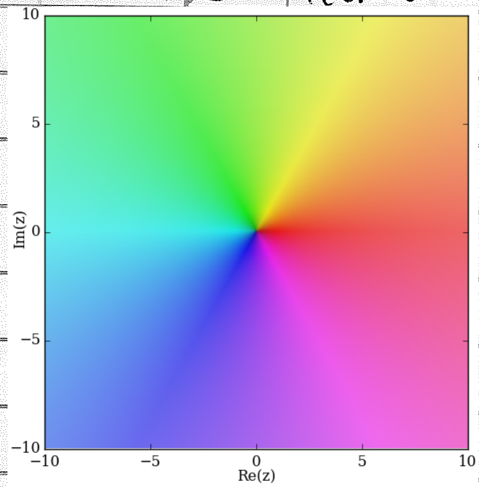
For a function $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = \sim \end{cases}$, we again may form ordered pairs $(z, f(z))$ BUT z is two-dimensional (lies in a plane) and so is $f(z)$, SO we would need a four-dimensional graph...

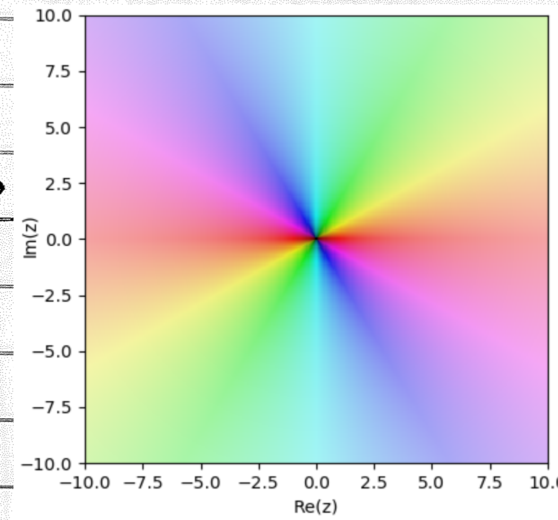
Domain coloring sidesteps the issue: imagine $z = re^{i\,\mathrm{Arg}(z)}$ ~ we use bright/dark to encode large/small values of r and we use a color wheel for the argument:



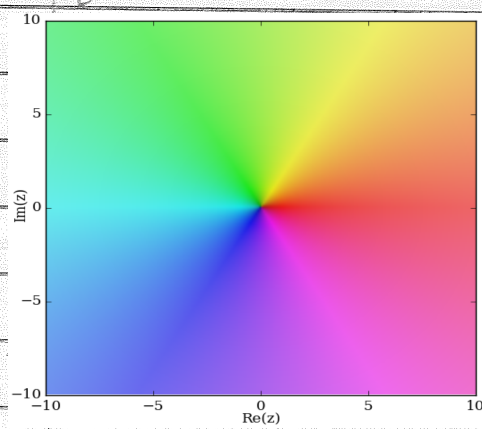
To color a function, we color at each position z the r and $\mathrm{Arg}(z)$ corresponding to $f(z)$.

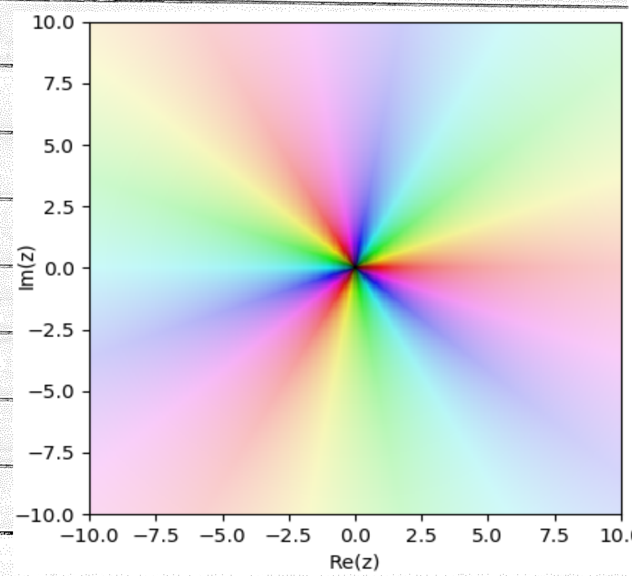
For example: $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = z^2 \end{cases}$



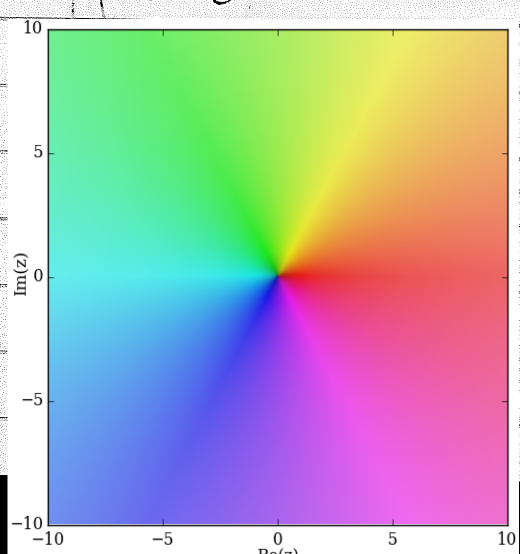


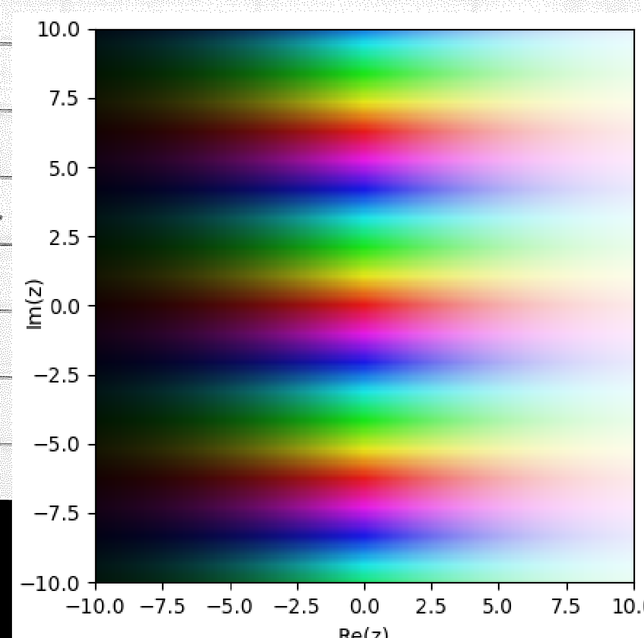
$\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = z^3 \end{cases}$





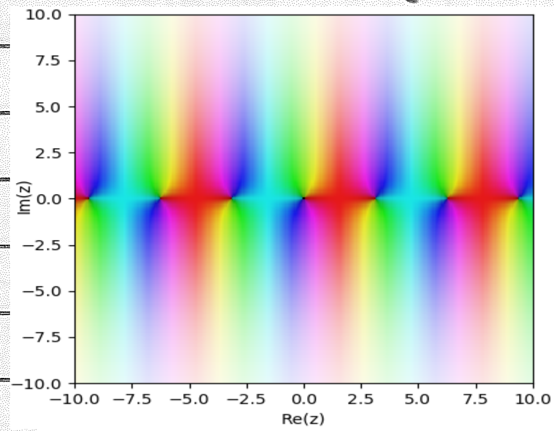
$\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = e^z \end{cases}$



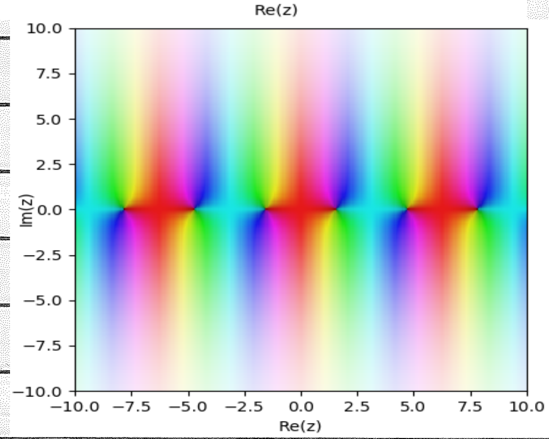


Usually, we drop the first picture + just show 2nd

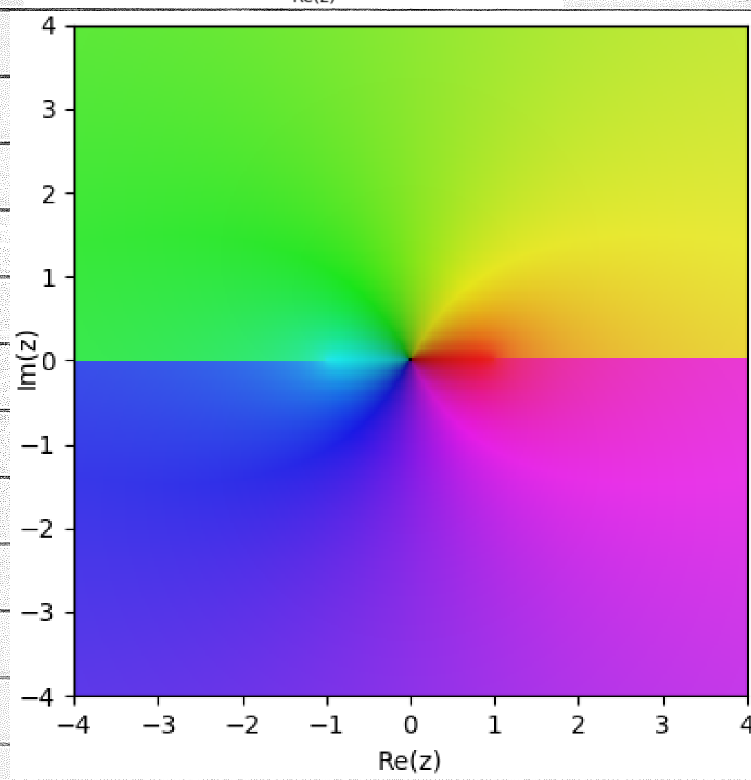
$$\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = \sin(z) \end{cases}$$



$$\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = \cos(z) \end{cases}$$



$$\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = \mathrm{Arcsin}(z) \end{cases}$$

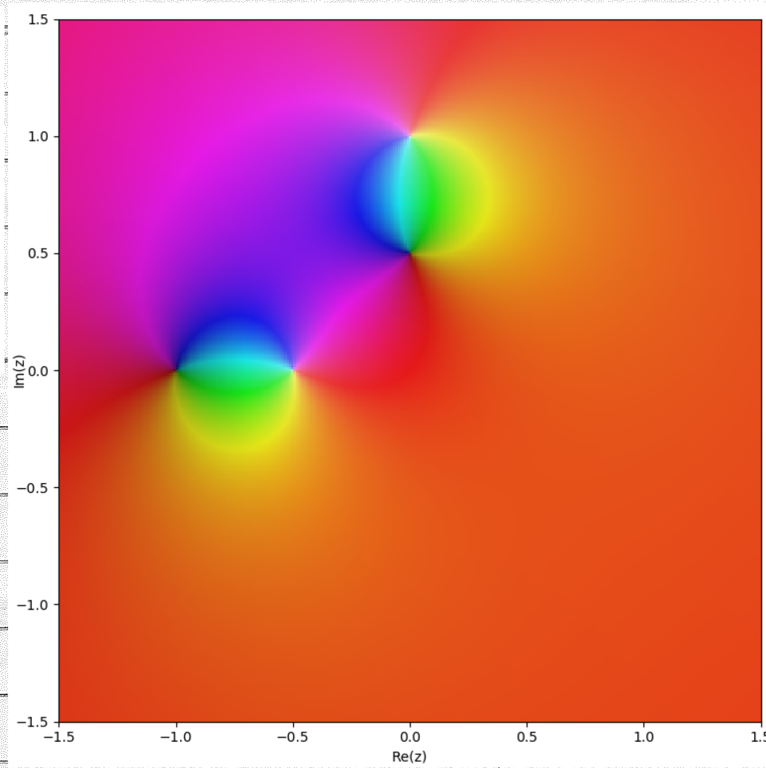


Zeros & blowups

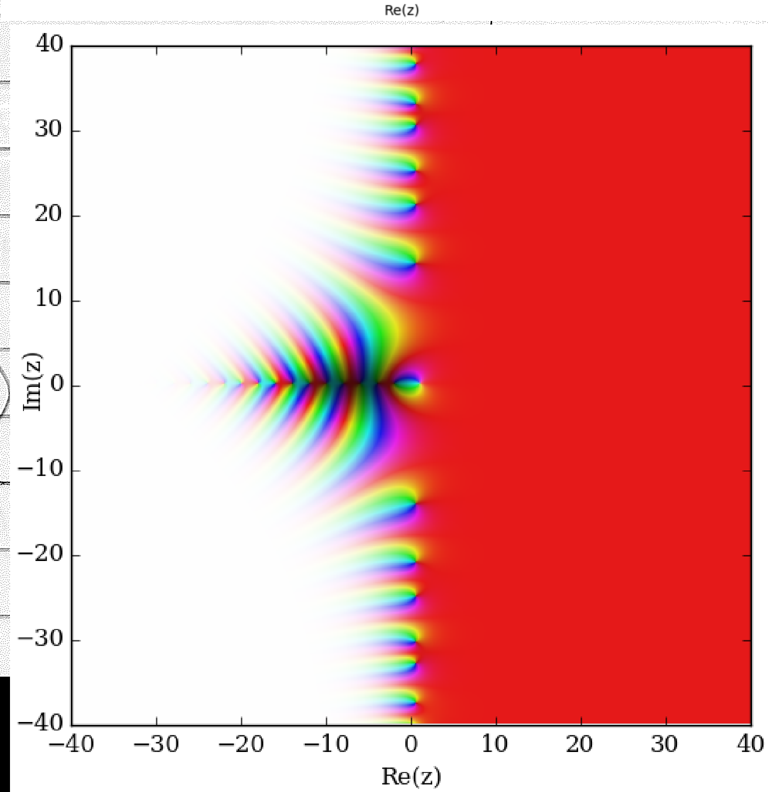
A zero is a place where $f(z)=0$.

A "blowup" is a place where $f(z)=\infty$.

Brightness $\longrightarrow$ zeros are white dots, blowups are black dots



$$\begin{cases} f: \mathbb{C} \to \overline{\mathbb{C}} \\ f(z) = \dfrac{(z-0.5i)(z+1)}{(z+0.5)(z-i)} \end{cases}$$



$$\begin{cases} f: \mathbb{C} \to \overline{\mathbb{C}} \\ f(z) = \zeta(z) \end{cases}$$

(Riemann zeta function)

Contours

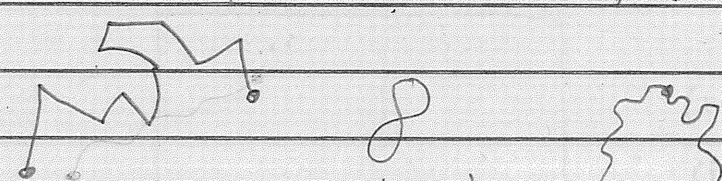
When we integrate in $\mathbb{C}$ we do so along curves called contours.

Def: An arc in $\mathbb{C}$ is a set of points

$$\{z(t) = x(t) + iy(t),\ a \le t \le b\}$$
$$x, y \in \mathbb{R}$$

An arc is called simple if there is no times t_1 and t_2 so that $z(t_1) = z(t_2)$. (no self-intersection).

We say an arc is a simple closed curve if it is simple with the exception that $z(a) = z(b)$. (a loop).



simple arc — arc but not simple arc — simple closed curve

Ex: Draw the arc $z(t) = \begin{cases} t - it, & 0 \le t \le 1 \\ 1 - i + t, & 1 \le t \le 2 \end{cases}$

Ex: Draw the simple closed curve $\begin{cases} z(t) = e^{it} \\ 0 \le t \le 2\pi \end{cases}$

Ex: Draw $\begin{cases} z(t) = e^{-it} \\ 0 \le t \le 2\pi \end{cases}$ (different!!)

Ex: Draw $\begin{cases} z(t) = e^{2it} \\ 0 \le t \le 2\pi \end{cases}$ (different!!)

Convention: "⊕ orientation" means "Counterclockwise"

An arc is called smooth if $z'(t) = x'(t) + iy'(t)$ exists, is continuous and is nonzero.

Arc length

if $\begin{cases} z(t) \\ a \le t \le b \end{cases}$ is an arc, then its length may be calculated as

$$L = \int_a^b |z'(t)|\, dt$$

Ex: Find length of arc $\begin{cases} z = t + it \\ 0 \le t \le 3 \end{cases}$

Soln: $z'(t) = 1 + i$

$$L = \int_0^3 |1+i|\, dt = \sqrt{2}(3-0) = 3\sqrt{2}$$

Ex: Find length of $\begin{cases} z = t^2 - 5t^2 i \\ 0 \le t \le 1 \end{cases}$

Soln: $z'(t) = 2t - 10ti$

$$|z'| = \sqrt{4t^2 + 100t^2} = \sqrt{104t^2} = \sqrt{104}\, t$$

So,

$$L = \int_0^1 \sqrt{104}\, t\, dt = \frac{\sqrt{104}}{2}\left[1 - 0\right] = \frac{\sqrt{104}}{2}$$

We define a contour to be piecewise smooth arc.
Similarly, a simple closed contour is a piecewise smooth simple closed arc.

Contour integrals

if C is a contour described by $\begin{cases} z(t) \\ a \le t \le b \end{cases}$, then the

contour integral of f along C is :

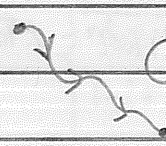
$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

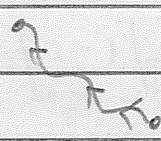
Some facts

for any $\alpha \in \mathbb{C}$,

* $$\int_C \alpha f(z)\,dz = \alpha \int_C f(z)\,dz$$

* $$\int_C f(z)+g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

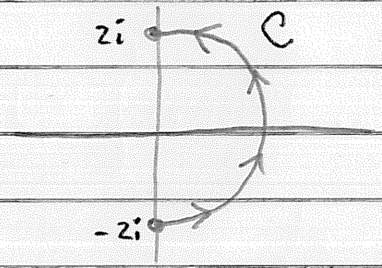
Given any contour C  C, there is another

contour $-C$  who has opposite orientation

* $$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

Ex: Compute $\int_C \bar{z}\,dz$ where $C = \begin{cases} 2e^{it} \\ -\frac{\pi}{2} \le t \le \frac{\pi}{2} \end{cases}$

Soln: First draw C:



We compute $z'(t) = 2ie^{it}$ and so

$$\int_C \bar{z}\,dz = \int_{-\pi/2}^{\pi/2} \overline{(2e^{it})} \cdot 2ie^{it}\,dt$$

$= 2e^{-it}$

$$= 4i \int_{-\pi/2}^{\pi/2} 1\,dt = 4i\left[t\right]_{-\pi/2}^{\pi/2} = 4i\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= 4\pi i$$

Ex: $\int_C z^2+1\,dz$, where C is unit circle

Ex: $\int_C \frac{1}{z}\,dz$, where C is unit circle

Ex: $\int_C \text{Im}(z) - \text{Re}(z) - 3\,\text{Re}(z)^2 i$, C is