More about logs

$$\log(z_1 z_2) = \ln(|z_1 z_2|) + i\arg(z_1 z_2)$$

(p.12)

$$= \underbrace{\ln(|z_1||z_2|)}_{\ln(|z_1|)+\ln(|z_2|)} + i(\arg(z_1) + \arg(z_2))$$

(calc' log props)

$$= [\ln(|z_1|) + i\arg(z_1)] + [\ln(|z_2|) + i\arg(z_2)]$$

$$= \log(z_1) + \log(z_2)$$

Similarly,

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$$

These are not necessarily still true for **Log**:

$$\mathrm{Log}(-1) = \pi i$$

but

$$\mathrm{Log}((-1)(-1)) = \mathrm{Log}(1) = 0, \text{ not } \pi i + \pi i = 2\pi i$$

We observe

$$\log(z^2) = \log(z \cdot z) = \log(z) + \log(z) = 2\log(z)$$

$$\log(z^3) = \ldots = 3\log(z)$$

$$\vdots$$

$$\log(z^n) = n\log(z)$$

So plugging both sides into e^z yields

$$z^n = e^{n\log(z)}$$

Since $z^{1/n}$ obeys $(z^{1/n})^n = z$, we can show after some work,

$$z^{1/n} = \exp\left(\tfrac{1}{n}\log(z)\right) \leftarrow \text{this is an expression for the } n^{th} \text{ roots of } z$$

Complex Exponents

When $z \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{C}$, then we define

$$z^c = e^{c\log(z)}$$

The principal value of z^c is defined by $(P.V.)z^c = e^{c\mathrm{Log}(z)}$

Ex: Compute P.V. i^i.

Soln: $(P.V.)i^i = \exp(i\,\mathrm{Log}(i))$

$$= \exp(i[\ln|i| + i\,\mathrm{Arg}(i)])$$

$$= \exp(i[\underbrace{\ln(1)}_{=0} + i(\tfrac{\pi}{2})])$$

$$= \exp(-\tfrac{\pi}{2}).$$

So the principal value of i^i is real... strange!!

Ex: Compute i^{-i}

Soln: $i^{-i} = \exp(-i\log(i))$

$$= \exp(-i[0 + i[\tfrac{\pi}{2} + 2n\pi]])$$

$$= e^{\frac{\pi}{2} + 2n\pi}, n \in \mathbb{Z}$$

So far we have: polynomials, exp, log

More complex functions

Sin and cos functions

From Euler formula for $\theta \in \mathbb{R}$:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

we obtain

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$\overset{\text{cos even}}{\underset{\text{sin odd}}{=}} \cos(\theta) - i\sin(\theta)$$

So,

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \qquad e^{i\theta} - e^{-i\theta} = 2i\sin(\theta)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Suggests the following definition for complex trig functs:

Def: $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ are defined by

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

All of trigonometry can follow from these defs.

Ex: Show $\frac{d}{dz}\sin(z) = \cos(z)$

Ex: Show $\sin^2(z) + \cos^2(z) = 1$

Ex: Show $\sin(i) = \left(\frac{e}{2} - \frac{1}{2e}\right)i$

Sinh and cosh functions

Similar to trig

Def: $\begin{cases} \sinh: \mathbb{C} \to \mathbb{C} \\ \sinh(z) = \dfrac{e^{z} - e^{-z}}{2} \end{cases}$ $\begin{cases} \cosh: \mathbb{C} \to \mathbb{C} \\ \cosh(z) = \dfrac{e^{z} + e^{-z}}{2} \end{cases}$

Ex: Show $\frac{d}{dz}\cosh(z) = \sinh(z)$

Ex: Show $\sinh(iz) = i\sin(z)$

arcsin and arccos

Let $z = \sin(w)$ so $w = \underbrace{\arcsin(z)}_{\text{what is it?}}$.

By def,

$$z = \sin(w) = \frac{e^{iw} - e^{-iw}}{2i}$$

$$\Rightarrow \quad 2iz = e^{iw} - e^{-iw}$$

Let $v = e^{iw}$, so $\frac{1}{v} = e^{-iw}$ (negative exponent → reciprocal)

This yields

$$2iz = v - \frac{1}{v}$$

or

$$0 = v^2 - 2izv - 1 \overset{QF}{\Longrightarrow} v = \frac{2iz \pm \sqrt{(-2iz)^2 - 4(-1)}}{2}$$

Taking the ⊕ soln yields

$$v = iz + \frac{1}{2}\sqrt{-4z^2+4}$$

$$= iz + \frac{1}{2}2\sqrt{-z^2+1}$$

$\sqrt{-1} = i$ $$= iz + \sqrt{1-z^2}$$

But $v = e^{iw}$, so we have

$$e^{iw} = iz + \sqrt{1-z^2}$$

$$w = \arcsin(z) = \frac{1}{i}\log\left(iz + \sqrt{1-z^2}\right)$$

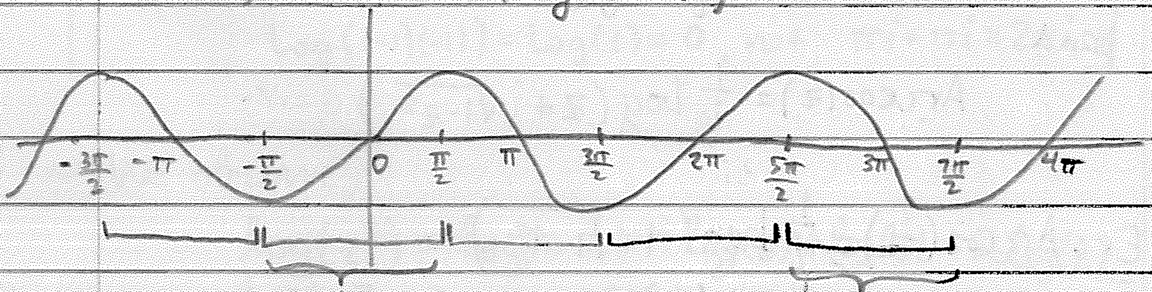
↑ arcsin is MULTIVALUED!!

*note

$\sqrt{}$ denotes principal square root!!

ie $\sqrt{z} = e^{\frac{1}{2}\mathrm{Log}(z)}$

The multivalued aspect of arcsin is not surprising in hindsight. From trigonometry:



one-to-one piece usually used to restrict

could pick any one-to-one piece like this

$\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

to get the "usual" inverse

$\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin : [\frac{5\pi}{2}, \frac{7\pi}{2}] \to [-1,1]$

to get a different

$\arcsin : [-1,1] \to [\frac{5\pi}{2}, \frac{7\pi}{2}]$

These different inverses come about by the multivalued aspect of our formula!!

Ex:

$$\arcsin(1) = \frac{1}{i}\log\left(i+\sqrt{1-1^2}\right)$$

$$= \frac{1}{i}\log(i)$$

$$= \frac{1}{i}\log\left(e^{\pi/2\, i}\right)$$

$$= \frac{1}{i}\left[\underbrace{\ln(\underbrace{|e^{\pi/2\, i}|}_{=1})}_{=0} + i\left(\mathrm{Arg}\left(e^{\pi/2\, i}\right)+2n\pi\right)\right],\ n\in\mathbb{Z}$$

$$= \frac{1}{i}\left[i\left(\frac{\pi}{2}+2n\pi\right)\right]$$

$$= \frac{\pi}{2}+2n\pi\ ,\ n\in\mathbb{Z}$$

We define principal arcsin and arccos by

$$\mathrm{Arcsin}(z) = \frac{1}{i}\,\mathrm{Log}\left(iz+\sqrt{1-z^2}\right)$$

and

$$\mathrm{Arccos}(z) = \frac{1}{i}\,\mathrm{Log}\left(z+\sqrt{z^2-1}\right)$$

Ex: Arccos(-1) [redacted] $\sqrt{2}+i=$

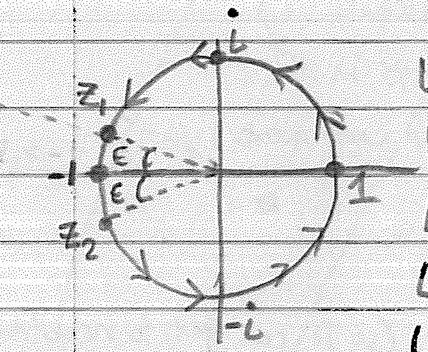
Branch Cuts

We have numerous multivalued functions: log, arcsin, arccos
and their principal "branches" : Log, Arcsin, Arccos

Def: A branch cut is a set in $\mathbb{C}$ which is used to define a "branch" (i.e. single-valued) of a multivalued function.

We have defined $-\pi < \operatorname{Arg}(z) \leq \pi$ and this restriction is what we used to define Log. This gave us a branch cut on the negative real axis for Log.

Let us observe the discontinuity directly:



$\operatorname{Log}(1) = 0$

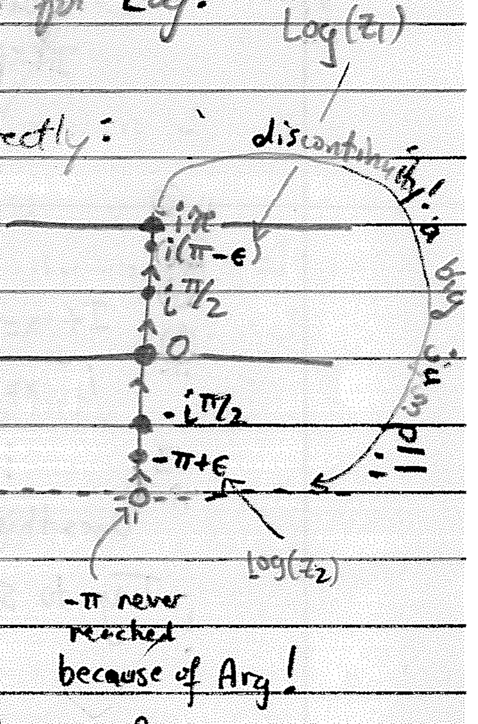
$\operatorname{Log}(i) = i\frac{\pi}{2}$

$\operatorname{Log}(z_1) = i(\pi - \epsilon)$

$\operatorname{Log}(-1) = i\pi$

$\operatorname{Log}(z_2) = i(-\pi + \epsilon)$

$\operatorname{Log}(-i) = -i\frac{\pi}{2}$



-π never reached because of Arg!

Since we define z^c, Arcsin, and Arccos in terms of Log, we can analyze their branch cuts as well.

Branch cut of z^c

Simple:

$$z^c = \exp(c\,\mathrm{Log}(z))$$

and so we get immediately from the branch cut of Log the same cut

Branch cut of Arcsin

$$\mathrm{Arcsin}(z) = \frac{1}{i}\mathrm{Log}\left(iz + \overset{=(1-z^2)^c}{\overset{\downarrow}{\sqrt{1-z^2}}}\right)$$

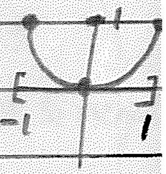
Two ways we could get the cut:

① from Log when $iz + \sqrt{1-z^2} \leq 0$

or

② from $\sqrt{1-z^2}$ when $1-z^2 < 0$

We will argue that ① can't ever happen and ② happens when $z \in (-\infty, 1) \cup (1, \infty)$.

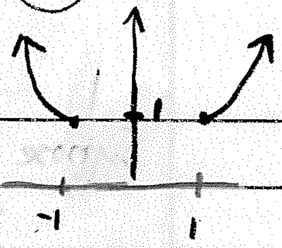


First, if $-1 \leq z \leq 1$ then $0 \leq z^2 \leq 1$

$0 \geq -z^2 \geq -1$

$1 \geq 1 - z^2 \geq 0$

Therefore $\sqrt{1-z^2} \in [0, \infty)$ and $iz + \sqrt{1-z^2} \in \mathbb{C} \setminus \mathbb{R}$ and so neither ① or ② can happen.



If $-\infty<z<-1$ or $1<z<\infty$, then $1<z^2<\infty$

$$\downarrow$$

$$-1>-z^2>-\infty$$

$$0>1-z^2>-\infty$$

So $1-z^2<0$ and ② satisfied. So these points are on the branch cut.

It remains to show that if $z\in\mathbb{C}\setminus\mathbb{R}$, then neither ① nor ② hold. Let $z=x+iy$ with $y\neq 0$.

Firstly, ② does not hold because $z^2=(x+iy)^2$
$= x^2+2xyi-y^2$
and so $1-z^2=(1-x^2+y^2)-2xyi\notin(-\infty,0)$, since it is either complex (when $x\neq 0$) or positive (if $x=0$).

It remains to show that ① fails for $z\in\mathbb{C}\setminus\mathbb{R}$.

If we had a z so that $iz+\sqrt{1-z^2}\in(-\infty,0)$, then $\sqrt{1-z^2}=-iz-a$ for some $a>0$.

b/c then
$iz+\sqrt{1-z^2}$
$\shortparallel$
$iz-iz-a=-a<0$

$$\downarrow$$

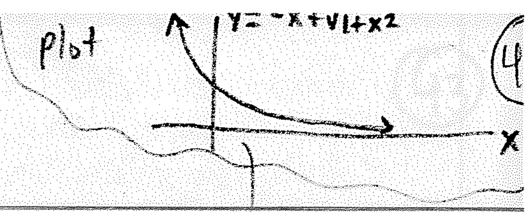
$$(\sqrt{1-z^2})^2=(-iz-a)^2$$

$$1-z^2=i^2z^2+2iza+a^2$$

$$1-\cancel{z^2}=-\cancel{z^2}+2iza+a^2$$

$$\Rightarrow z=\frac{1-a^2}{2ia}=-i\overbrace{\left(\frac{1}{2a}-\frac{a}{2}\right)}^{\in\mathbb{R}}$$

In other words, z would have to lie on y-axis.



Suppose $z = ib$, $b \in \mathbb{R}$. Then

$$iz + \sqrt{1-z^2} = i(ib) + \sqrt{1-(ib)^2}$$
$$= -b + \sqrt{1+b^2}$$

But this quantity is never negative! Therefore ① fails here.

To summarize

Where z lives	Does ① happen?	Does ② happen?
$-1 \leq z \leq 1$	NO	NO
$z \geq 1$	doesn't matter	Yes
$z \leq -1$	doesn't matter	YES
$z \in \mathbb{C} \setminus \mathbb{R}$	NO	NO

Therefore the branch cut of Arcsin is $(-\infty,-1) \cup (1,\infty)$

recall: $f: X \to Y$ one-to-one means $f(x)=f(y) \to x=y$

$f: X \to Y$ onto means $\forall y \in Y \exists x \in X \, (f(x)=y)$

Möbius transformations ("linear fractional transformation")

note: they form a group under composition!

Consider the functions

$$\begin{cases} f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \\ f(z) = \dfrac{az+b}{cz+d}, \quad ad-bc \neq 0 \end{cases}$$

Note: here we are on Riemann sphere $\overline{\mathbb{C}}$ (see p.24)

so we make sense of

$$f(\infty) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c}$$

and $f\left(\frac{-d}{c}\right) = \infty$

Theorem: A Möbius transformation is one-to-one and onto $\overline{\mathbb{C}}$.

Proof: Pick any $w \in \overline{\mathbb{C}}$, we will show there is exactly one $z \in \overline{\mathbb{C}}$ so that $f(z) = w$.

$$w = f(z) = \frac{az+b}{cz+d}$$

$$w(cz+d) = az+b$$

$$wcz + wd = az+b$$

$$z(wc-a) = b-wd$$

$$z = \frac{b-wd}{wc-a}. \ \blacksquare$$

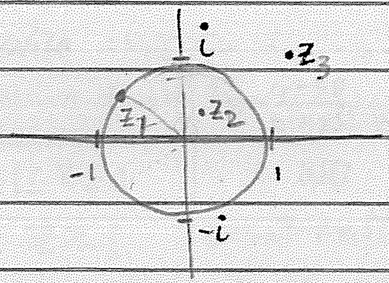
Also this shows if $f(z) = \dfrac{az+b}{cz+d}$, then $f^{-1}(z) = \dfrac{-dz+b}{cz-a}$

Ex: Consider the Möbius transformation $f(z)=\frac{1}{z}$

What does it do?

Consider $z=re^{i\theta}$, then $f(z)=\frac{1}{r}e^{-i\theta}$

What does f do to unit circle?



① For points on unit circle — z_1 — $z_2 = 1e^{i\theta_1}$

$$\downarrow f$$

$$f(z_2)=e^{-i\theta_1}$$

in other words, it reflects across x-axis

circle ⟶ itself!

② for points inside circle — z_1 — $z_2 = re^{i\theta}$, $r<1$

$$\downarrow f \qquad \downarrow$$

$$f(z_2)=\frac{1}{r}e^{-i\theta} \qquad \frac{1}{r}>1$$

in other words, reflects and extends outside circle

interior of circle ⟶ exterior of circle

③ for points outside circle — z_3 — $z_3 = re^{i\theta}$, $r>1$

$$\downarrow f \qquad \downarrow$$

$$f(z_3)=\frac{1}{r}e^{-i\theta} \qquad \frac{1}{r}<1$$

In other words, reflects + pulls inside

exterior ⟶ interior

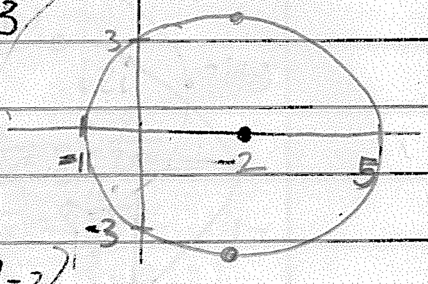
Recall: $|z| = |x+iy| = \sqrt{x^2+y^2}$

$z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$

$|z-c| = R$ ~ circle centered at C w/ radius R

Ex: What does $f(z) = \frac{1}{z}$ do to a different circle?

Consider the circle $|z-2| = 3$



Call this circle $K = \{z \in \mathbb{C} : |z-2i| = 3\}$

We want to understand what f does to K, i.e. what is f(K)?

So,

$$w \in f(K) \longleftrightarrow \frac{1}{w} \in K$$

$$\longleftrightarrow \left|\frac{1}{w} - 2\right| = 3$$

Multiply by $|w|$ to get

$$|1-2w| = 3|w|$$

Square both sides

$$|1-2w|^2 = 9|w|^2$$

Using $z\bar{z} = |z|^2$ and $\overline{1-2w} = \bar{1} - \overline{2w} = 1 - 2\bar{w}$,

$$(1-2w)(1-2\bar{w}) = 9w\bar{w}$$

$$1 - 2\bar{w} - 2w + 4w\bar{w} = 9w\bar{w}$$

$$-5w\bar{w} - 2\bar{w} - 2w = -1$$

$$w\bar{w} + \tfrac{2}{5}\bar{w} + \tfrac{2}{5}w = \tfrac{1}{5}$$

$$w\bar{w} + \tfrac{2}{5}\bar{w} + \tfrac{2}{5}w + \left(\tfrac{2}{5}\right)^2 = \tfrac{1}{5} + \tfrac{4}{25}$$

$\alpha = \frac{2}{5}$

$$w\left(\bar{w} + \tfrac{2}{5}\right) + \left(\tfrac{2}{5}\right)\left(\bar{w} + \tfrac{2}{5}\right) = \tfrac{9}{25}$$

$$\left(w + \tfrac{2}{5}\right)\left(\bar{w} + \tfrac{2}{5}\right) = \tfrac{9}{25}$$

$$\left|w - \left(-\tfrac{2}{5}\right)\right| = \tfrac{9}{25} \leftarrow \text{circle centered at } -\tfrac{2}{5} \text{ of radius } \tfrac{9}{25}$$

Alg trick

To factor $w\bar{w} + \alpha\bar{w} + \alpha w = \beta$ add α^2 to both sides

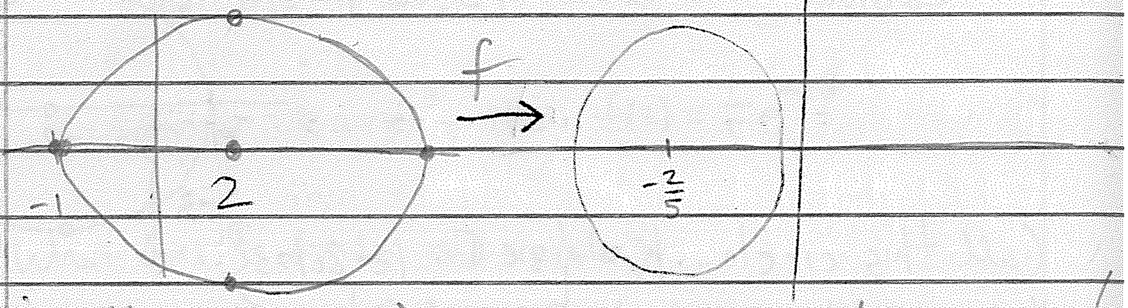
Algebra trick

To factor

$w\bar{w}$

Recall: $z = \underset{\substack{\uparrow \\ \text{Re}\,z}}{x} + \underset{\substack{\uparrow \\ i\,\text{Im}\,z}}{iy} \Rightarrow z + \bar{z} = (x+iy)+(x-iy)$
$= x = \text{Re}(z)$

So $f(z) = \frac{1}{z}$ mapped



in other words $\longrightarrow$ a circle mapped onto a circle!

Ex: What does $K = \{z \in \mathbb{C} : |z-1| = 1\}$ map to?

As before $w \in f(K) \leftrightarrow \frac{1}{w} \in K$

$\leftrightarrow \left|\frac{1}{w} - 1\right| = 1$

z

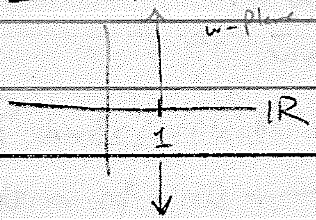
$|1-w|^2 = |w|^2$

$(1-w)(1-\bar{w}) = w\bar{w}$

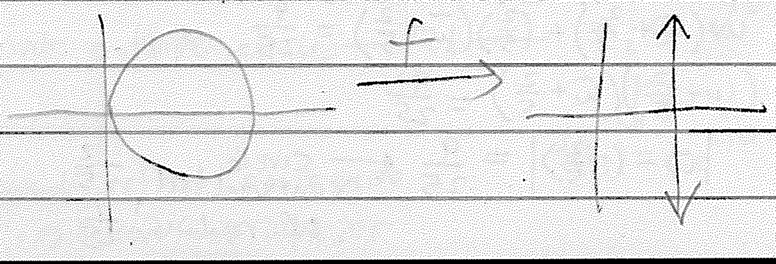
What is this? $1 - \bar{w} - w + w\bar{w} = w\bar{w}$

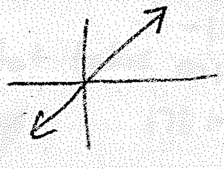
It's the vertical $1 = w + \bar{w}$

line at 1: iR $\longrightarrow 1 = \text{Re}(w)$



So, a circle mapped to a line!





Ex: Let $K=\{z\in\mathbb{C} : z = x+ix, -\infty<x<\infty\}$

What is $f(K)$?

If $z=x+ix$, then

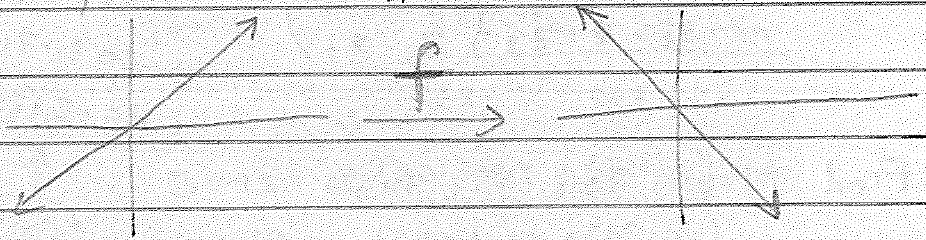
$$f(z)=\frac{1}{z}=\frac{1}{x+ix}=\frac{x-ix}{x^2+x^2}=\frac{x}{2x^2}-i\frac{x}{2x^2}$$

$$=\left(\frac{1}{2x}\right)-i\left(\frac{1}{2x}\right)$$

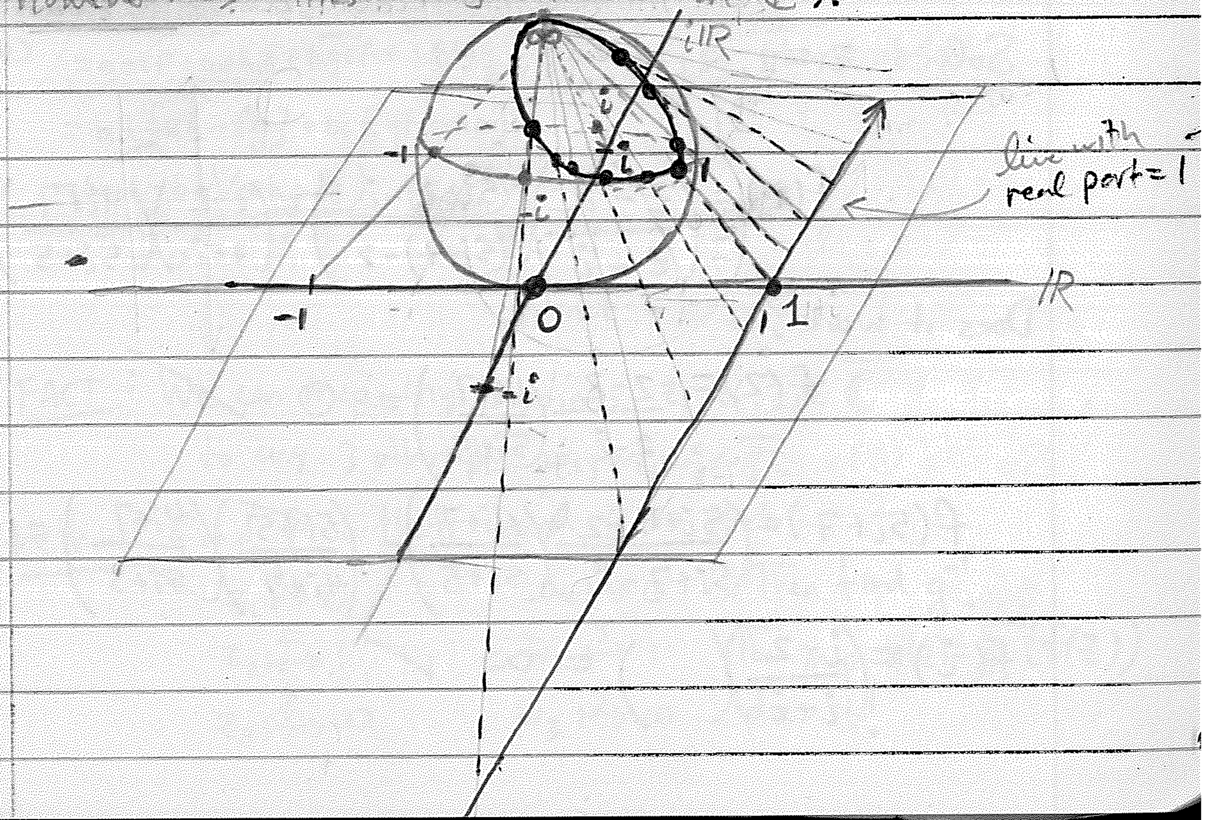
call $s=\frac{1}{2x}$

describes a line again! $\longrightarrow = s-is$

So, a line was mapped to a line



However → "lines" are just circles on $\overline{\mathbb{C}}$!!



Theorem: Möbius transformations map circles in $\overline{\mathbb{C}}$ to circles in $\overline{\mathbb{C}}$.

Theorem: Given any three distinct points $z_1, z_2, z_3 \in \overline{\mathbb{C}}$, there is a unique Möbius transformation f that obeys $f(z_1)=0$, $f(z_2)=1$, and $f(z_3)=\infty$.

Moreover, it is given by

$$f(z) = \frac{z-z_1}{z-z_3}\left(\frac{z_2-z_3}{z_2-z_1}\right) \longleftarrow$$

this is Möbius with
$a = (z_2-z_3)$,
$b = -z_1(z_2-z_3)$
$c = z_2-z_1$
$d = -z_1(z_2-z_1)$

$\uparrow$ Can show satisfies $ad-bc \neq 0$ Conditi

Ex: Find Möbius trans. that maps $2 \mapsto 0$
$5i+7 \mapsto 1$
and $i \mapsto \infty$

Soln: $z_1 = 2$, $z_2 = 5i+7$, $z_3 = i$

$$\downarrow$$

$$f(z) = \left(\frac{z-2}{z-i}\right)\left(\frac{(5i+7)-i}{(5i+7)-2}\right) = \left(\frac{z-2}{z-i}\right)\left(\frac{4i+7}{5i+5}\right)$$

Does it work?

$$f(2) = \frac{2-2}{2-i}\left(\frac{4i+7}{5i+5}\right) = 0 \checkmark$$

$$f(5i+7) = \left(\frac{5i+7-2}{5i+7-i}\right)\left(\frac{4i+7}{5i+5}\right) = \left(\frac{\cancel{5i+5}}{\cancel{4i+7}}\right)\left(\frac{\cancel{4i+7}}{\cancel{5i+5}}\right) = 1 \checkmark$$

$$f(i) = \left(\frac{i-2}{i-i}\right)\left(\quad\right) = \infty \checkmark$$

EX: Find Möbius transformation that takes $-1 \mapsto 0$, $\infty \mapsto 1$, $17i \mapsto \infty$

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Theorem: A composition of Möbius transformation is a Möbius transformation.

$ad-bc\neq 0$ $\quad$ $eh-fg\neq 0$

Pf: $f(z)=\dfrac{az+b}{cz+d}$ $\quad$ $g(z)=\dfrac{ez+f}{gz+h}$

$$(f\circ g)(z)=f(g(z))=\frac{a\left(\frac{ez+f}{gz+h}\right)+b}{c\left(\frac{ez+f}{gz+h}\right)+d}$$

mult by $\dfrac{gz+h}{gz+h}$

$$=\frac{aez+fa+bgz+bh}{cez+cf+dgz+hd}$$

$$=\frac{(ae+bg)z+(fa+bh)}{(ce+dg)z+(cf+hd)}$$

and $(ae+bg)(cf+hd)-(fa+bh)(ce+dg)$

$=\cancel{aecf}+aehd+bgcf+\cancel{bghd}-\cancel{face}-fadg-bhce-\cancel{bhdg}$

$=ad[eh-fg]+bc[gf-he]$

$=\underbrace{(ad-bc)}_{\neq 0}\underbrace{(eh-fg)}_{\neq 0}\neq 0$. $\blacksquare$

FACT: We can map any 3 $\overset{\text{distinct}}{\wedge}$ points $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ to any 3 other pts $w_1, w_2, w_3 \in \overline{\mathbb{C}}$.

Method: find f so that and g so that

$z_1 \xrightarrow{f} 0$ $\quad$ $w_1 \xrightarrow{g} 0$ $\quad$ Then find g^{-1}.

$z_2 \xrightarrow{f} 1$ $\quad$ $w_2 \xrightarrow{g} 1$ $\quad$ The map $h(z)=g^{-1}(f(z))$

$z_3 \xrightarrow{f} \infty$ $\quad$ $w_3 \xrightarrow{g} \infty$ $\quad$ does it!

㊼53

Ex: Find a Möbius transformation f that maps

$$1 \mapsto 3$$
$$-1+i \mapsto 7$$
$$\infty \mapsto 8$$

Soln: First find M.T. $\overset{f_1}{1 \mapsto 0}$

$$-1+i \mapsto 1 \quad \Longrightarrow \quad f_1(z) = \text{"}\frac{z-1}{z-\infty}\text{"}\left(\frac{-1+i-}{-1+i}\right.$$

$$\infty \mapsto \infty \qquad\qquad = \frac{z-1}{(i-2)}$$

Now find M.T. f_2 s.t.

$$3 \mapsto 0$$
$$7 \mapsto 1 \quad \Longrightarrow \quad f_2(z) = \frac{z-3}{z-8}\left(\frac{7-8}{7-3}\right)$$
$$8 \mapsto \infty$$

$$= \frac{z-3}{z-8}\left(\frac{-1}{4}\right)$$

$$w = f_2(z) = -\frac{1}{4}\left(\frac{z-3}{z-8}\right)$$

↓ find inverse by solving for z

$$-4w(z-8) = z-3$$

$$-4wz + 32w = z-3$$

$$z(-4w-1) = -3-32w$$

$$z = \frac{32w+3}{4w+1}$$

So $f_2^{-1}(z) = \dfrac{32z+3}{4w+1}$.

Now $f = f_2^{-1}(f_1(z)) = \dfrac{32\left(\frac{z-1}{i-2}\right)+3}{4\left(\frac{z-1}{i-2}\right)+1}$

$$= \frac{32z - 32 + 3i - 6}{4z - 4 + i - 2}$$

$$= \frac{32z + (-38+3i)}{4z + (-6+i)}$$

Did it work? (Notice: $32(-6+i) - 4(-38+3i) = -40+2i \neq 0$)

$$f(1) = \frac{32-38+3i}{4-6+i} = \frac{-6+3i}{-2+i} = 3\,\frac{(-2+i)}{(-2+i)} = 3 \checkmark$$

$$f(-1+i) = \frac{32(-1+i) + (-38+3i)}{4(-1+i) + (-6+i)} = \frac{-32+32i-38+3i}{-4+4i-6+i}$$

$$\begin{array}{r} {}^{1} \\ 32 \\ \underline{38} \\ 70 \end{array}$$

$$= \frac{35i-70}{5i-10}$$

$$= \frac{7(5i-10)}{5i-10} = 7 \checkmark$$

$$f(\infty) = \lim_{z \to \infty} \frac{32z + (-38+3i)}{4z + (-6+i)} = \frac{32}{4} = 8 \checkmark$$