<u>Recall</u>: Δ-inequality $|z_1+z_2| \leq |z_1|+|z_2|$

<u>Theorem</u>: Let $f, g : \mathcal{U} \subseteq \mathbb{C} \to \mathbb{C}$ be continuous at z_0. Then the sum $f+g : \mathcal{U} \to \mathbb{C}$ is continuous at z_0.

<u>Proof</u>: Suppose f and g are continuous at z_0. That means,

(A) for all $\epsilon > 0$ there is $\delta > 0$ s.t. $\left(0 < |z-z_0| < \delta \to |f(z)-f(z_0)| < \epsilon\right)$.

and

(B) for all $\epsilon > 0$ there is $\delta > 0$ s.t. $\left(0 < |z-z_0| < \delta \to |g(z)-g(z_0)| < \epsilon\right)$.

Let $\epsilon > 0$. By (A) there is some δ_A such that

if $|z-z_0| < \delta_A$, then $|f(z)-f(z_0)| < \frac{\epsilon}{2}$

and by (B), there is some δ_B s.t.

if $|z-z_0| < \delta_B$, then $|g(z)-g(z_0)| < \frac{\epsilon}{2}$.

Choose $\delta = \min\{\delta_A, \delta_B\}$. Calculate

$$
\begin{aligned}
|(f+g)(z) - (f+g)(z_0)| &= |(f(z)-f(z_0)) + (g(z)-g(z_0))| \\
&\overset{\Delta\text{-ineq.}}{\leq} |f(z)-f(z_0)| + |g(z)-g(z_0)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}
$$

which completes the proof. ■

[Similarly, the product, the quotient, compositions, and powers of continuous functions are also continuous.]

$|(az+b)-(az_0+b)| = |a(z-z_0)|$

Theorem: Any function $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = az+b,\ a \neq 0 \end{cases}$ is continuous.

Proof: Let $\epsilon > 0$ and let $z_0 \in \mathbb{C}$. Choose $\delta = \frac{\epsilon}{|a|}$. Compute

$$\begin{aligned} |f(z) - f(z_0)| &= |(az+b) - (az_0+b)| \\ &= |a(z-z_0)| \\ &= |a|\,|z-z_0| \\ &< |a| \frac{\epsilon}{|a|} \\ &= \epsilon, \end{aligned}$$

completing the proof. ■

Consequence: any polynomial is continuous!!

Ex: $f(z) = z^2$ is continuous because $z^2 = \underset{\text{ctn}}{z} \cdot \underset{\text{ctn}}{z}$

Differentiation of $\mathbb{C}$-valued functions

Def: A function f is called differentiable at z with derivative $f'(z_0)$ provided that the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. Sometimes we write $\frac{df}{dz}$ for the derivative.

Ex: Consider $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = z^2 \end{cases}$. Show that $f'(z) = 2z$.

Soln: Calculate

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{z^2} + 2zh + h^2 - \cancel{z^2}}{h}$$

$$= \lim_{h \to 0} 2z + h$$

using continuity of $2z+h$ $\longrightarrow$ $= 2z.$

FACT: basic ideas from calculus still hold here...

Sum Rule: $(f(z) + g(z))' = f'(z) + g'(z)$

Product Rule: $(f(z) \cdot g(z))' = f'(z)g(z) + f(z)g'(z)$

Quotient Rule: $\left(\frac{f(z)}{g(z)}\right)' = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$

Chain Rule: $(f(g(z)))' = f'(g(z))g'(z)$

Cauchy-Riemann Equations

Recall: partial derivatives "treat y as constant"

Ex: $$\frac{\partial}{\partial x}\left[x^2+3xy-y^2\right] = 2x+3y-0 = 2x+3y$$

"treat x as constant"

$$\frac{\partial}{\partial y}\left[x^2+3xy-y^2\right] = 0+3x-2y = 3x-2y$$

Theorem: Suppose that $f(z)=u(x,y)+iv(x,y)$, where $z=x+iy$, and Suppose that $f'(z_0)$ exists, where $z_0=x_0+iy_0$. Then, all first-order partial derivatives of u and v exist at (x_0,y_0) and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Also, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Ex: Earlier we showed if $f(z)=z^2$, then $f'(z)=2z$.
In the new light, $f(x+iy)=(x+iy)^2 = x^2+2xyi+i^2y^2$
$$= (x^2-y^2)+(2xy)i$$

meaning we have $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$.

Now note that
$$\frac{\partial u}{\partial x}=2x,\ \frac{\partial u}{\partial y}=-2y,\ \frac{\partial v}{\partial x}=2y,\ \text{and}\ \frac{\partial v}{\partial y}=2x.$$

And we have
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

note: the word "analytic" and the word "holomorphic" are also used to denote having a complex derivative

Now consider the Contrapositive.

Corollary: If $f=u+iv$ and both $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ do not hold at (x_0,y_0), then $f'(x_0+iy_0)$ does not exist.

Ex: Show that $\begin{cases} f:\mathbb{C}\to\mathbb{C} \\ f(z)=|z| \end{cases}$ is not differentiable everywhere.

Soln: First find u and v:

$$f(x+iy)=|x+iy|=\underbrace{\sqrt{x^2+y^2}}_{u(x,y)}+\underset{v(x,y)}{0}\,i$$

Calculate

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(x^2+y^2\right)^{1/2}=\frac{1}{2}\left(x^2+y^2\right)(2x)$$

while

$$\frac{\partial u}{\partial y}=0$$

So we have $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$, and we conclude f' does not exist

(except maybe when $x=0$... would require more work)

Ex: Show $f(z)=\bar{z}$ is not diff'bl.

Recall: Euler formula $e^{iy} = \cos(y) + i\sin(y)$

Exponential Function

if $z = x+iy$, then we define

$$e^{z} := e^{x}e^{iy}$$

Theorem: $\frac{d}{dz}e^{z} = e^{x}e^{iy}$

Proof: Using euler formula,

$$e^{z} = e^{x}e^{iy} = \underbrace{e^{x}\cos(y)}_{u(x,y)} + i\underbrace{e^{x}\sin(y)}_{v(x,y)}$$

And by Theorem on p.30,

$$\frac{d}{dz}e^{z} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$$

So calculating

$$\frac{\partial u}{\partial x} = e^{x}\cos(y) \quad \text{and} \quad \frac{\partial v}{\partial x} = e^{x}\sin y$$

$$\frac{\partial u}{\partial y} = -e^{x}\sin y \quad \text{and} \quad \frac{\partial v}{\partial y} = e^{x}\cos y,$$

we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. So by Theorem on p.30,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^{x}\cos(y) + ie^{x}\sin(y)$$

$$= e^{x}e^{iy}$$

$$= e^{z},$$

as was to be shown. ∎

$$r\,e^{i\theta}$$

Notice that $e^z = e^x e^{iy}$ is already in polar form, telling us

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi i \quad \text{for } n \in \mathbb{Z}$$

So in particular, we know that e^z is $2\pi i$-periodic, i.e.

$$e^z = e^{z+2\pi i}$$

because

$$e^{z+2\pi i} = e^{x+(y+2\pi)i} = e^x e^{(y+2\pi)i}$$

$$= e^x\left[\cos(y+2\pi) + i\sin(y+2\pi)\right]$$

sin and cos are 2π-periodic $\longrightarrow$ $= e^x\left[\cos(y) + i\sin(y)\right]$

$$= e^z$$

note: sometimes we will write $\exp(z)$ for e^z

Ex: Show that $\exp\left(\frac{3+\pi i}{6}\right) = \frac{\sqrt{e}}{2}\left(\sqrt{3}+i\right)$

Ex: Show that $\exp(\bar{z})$ is not analytic anywhere.

Logarithm

Logarithms are typically understood as "the" inverse of the exponential, i.e.

$$\text{if} \quad z = e^w, \quad \text{then} \quad \log(z) = w$$

However we have a problem for complex logarithms:

since e^z is $2\pi i$-periodic, we in fact have for any $n \in \mathbb{Z}$,

$$z = e^w = e^{w + 2n\pi i} \Rightarrow \underbrace{\log(z) = w + 2n\pi i}$$

"multi-valued"... like arg!!

<u>Def</u>: We define the notation

$$\log(z) = \ln(|z|) + i\arg(z)$$

to be the "multi-valued logarithm".

<u>Note</u>: technically $\log : \mathbb{C}\setminus\{0\} \to \text{PowerSet}(\mathbb{C})$, but we usually think of it as "taking multiple values" as an abuse of notation

<u>Def</u>: The principal branch of log is defined by

$$\text{Log}(z) = \ln(|z|) + i\,\text{Arg}(z)$$

<u>Ex</u>: $\log(1) = \ln(|1|) + i\arg(1)$ $\qquad 1 = e^0 = e^{2\pi i} = e^{4\pi i} = \dots$

$= 0 + 2n\pi i, \; n \in \mathbb{Z}$

$\text{Log}(1) = \ln(|1|) + i\,\text{Arg}(1)$

$= 0 + 0i = 0$

$$-1 = e^{\pi i} = e^{3\pi i} = \dots$$

Ex: $\log(-1) = \ln(|-1|) + i \arg(-1)$

$= 0 + n\pi i, \quad n \in \mathbb{Z}$ and n odd

(OR)

$= 0 + (2m+1)\pi i, \quad m \in \mathbb{Z}$

$\mathrm{Log}(-1) = \ln(|-1|) + i\,\mathrm{Arg}(-1)$

$= 0 + \pi i$

Ex: $\log(-1-i) = \ln(|-1-i|) + i \arg(-1-i)$

$= \ln(\sqrt{2}) + i\left(-\frac{3\pi}{4} + 2n\pi\right)$

$\mathrm{Log}(-1-i) = \ln(|-1-i|) + i\left(-\frac{3\pi}{4}\right)$

$= \ln(\sqrt{2}) - \frac{3\pi}{4} i$