Limit Theorem 2: Suppose that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$.

Then

(i) $\lim_{z\to z_0} f(z)+g(z) = w_0+w_1$,

(ii) $\lim_{z\to z_0} f(z)g(z) = w_0 w_1$, and

(iii) if $w_1 \neq 0$, $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$

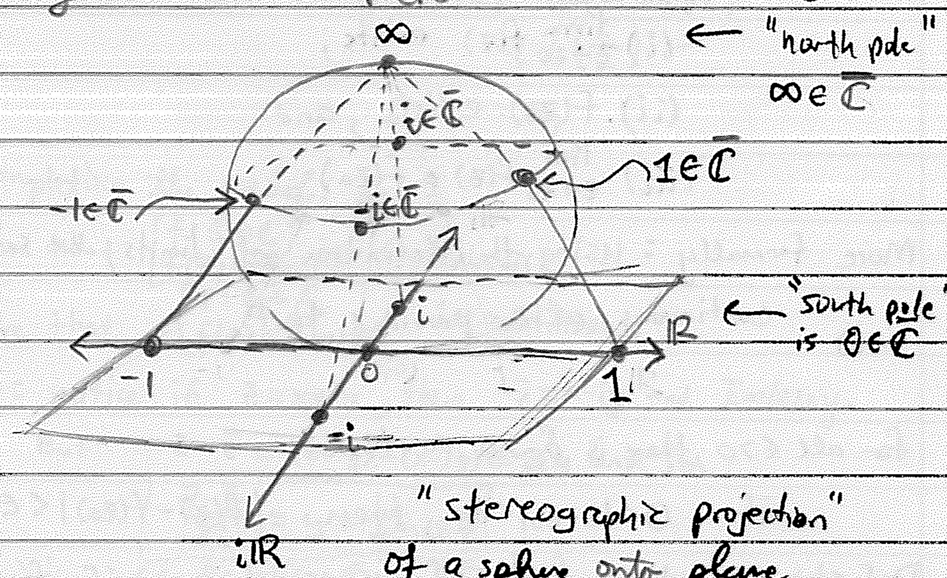
Ex: Prove that $\lim_{z\to 2i} \frac{z}{1+z} = \frac{2i}{1+2i}$. Prove that $\lim_{z\to 1+i} z^2+z = i$

Limits involving ∞

In $\mathbb{C}$ we do not distinguish between "$+\infty$" and "$-\infty$" — we take ∞ to be a singular "unbounded amount" that can be reached in any direction.

Riemann Sphere

Sometimes it is convenient to imagine ∞ as an element of the set of complex numbers. We visualize this using the Riemann sphere which we denote by $\overline{\mathbb{C}}$



"stereographic projection" of a sphere onto plane

Limit theorem with ∞: If $z_0, w_0 \in \mathbb{C}$, then

(i) $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$,

(ii) $\lim_{z \to \infty} f(z) = w_0$ if and only if $\lim_{z \to 0} f(\frac{1}{z}) = w_0$, and

(iii) $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$.

In short: We take "$\frac{1}{0} = \infty$" and "$\frac{1}{\infty} = 0$" somewhat literally in complex analysis!!

Ex: Use the above theorem to explain why $\lim_{z \to -1} \frac{iz+3}{z+1} = \infty$,

$\lim_{z \to \infty} \frac{2z+i}{z+1} = 2$, and

$\lim_{z \to \infty} \frac{2z^3 - 1}{z^2+1} = \infty$

Continuous functions

Def: A function f is called continuous at z_0 provided the following three things hold:

(i) $\lim_{z \to z_0} f(z)$ exists,

(ii) $f(z_0)$ exists, and

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

More formally: using the definition of limits, we say f is continuous at z_0 provided that

for all $\epsilon > 0$, there is $\delta_\epsilon > 0$ so that

if $0 < |z - z_0| < \delta_\epsilon$, then $|f(z) - f(z_0)| < \epsilon$.

Def: We say that $f: \mathcal{U} \to \mathbb{C}$ is continuous if for all $a \in \mathcal{U}$, f is continuous at a.

Ex: Prove that the function $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = 2z - 1 \end{cases}$ is continuous.

Scratch work: Need to find δ_ϵ so that for any $z_0 \in \mathbb{C}$,

$$|z - z_0| < \delta_\epsilon \Rightarrow |\underbrace{(2z-1)}_{f(z)} - \underbrace{(2z_0 - 1)}_{f(z_0)}| < \epsilon$$

- Start with conclusion:

$$|(2z-1) - (2z_0-1)| = |2z - 2z_0| = \underbrace{2}_{\text{forced!}} \overbrace{\underbrace{|z - z_0|}_{\text{so pick this ensuring it is } < \frac{\epsilon}{2}}}^{\text{Need}} < \epsilon$$

Proof: Let $\epsilon > 0$ and choose $\delta_\epsilon = \frac{\epsilon}{2}$. If $|z - z_0| < \frac{\epsilon}{2}$, then

Compute

$$\begin{aligned} |f(z) - f(z_0)| &= |(2z-1) - (2z_0 - 1)| \\ &= |2(z - z_0)| = 2|z - z_0| \\ &< 2\left(\frac{\epsilon}{2}\right) \\ &= \epsilon, \end{aligned}$$

completing the proof. ∎

Ex: Prove that $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = \bar{z} \end{cases}$ is continuous.

Ex: Prove that $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = \mathrm{Im}(z) \end{cases}$ is continuous.

hint: notice if $z = x + iy$, then $\bar{z} = x - iy$ and $\mathrm{Im}(z) = y$.

But $z - \bar{z} = (x+iy) - (x - iy) = 2iy$ and so

$$\boxed{\frac{z - \bar{z}}{2i} = y = \mathrm{Im}(z)}$$

Ex: Prove that $\begin{cases} f: \mathbb{C} \setminus \{\frac{1}{2}\} \to \mathbb{C} \\ f(z) = \frac{1}{1 - 2z} \end{cases}$ is continuous.