Complex Arithmetic in Polar Form

Usual rules of multiplication work:

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_1 + i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

in other words: multiplying two complex numbers multiplies their moduli and adds their arguments!!

<u>note</u>: this also implies $|z_1 z_2| = |z_1|\,|z_2|$

<u>Ex</u>: Write in polar form + multiply $z_1 = -2$ and $z_2 = -3$.

Clearly we should get $z_1 z_2 = 6$.

But,

$$z_1 = 2e^{i\pi} \quad \text{and} \quad z_2 = 3e^{i\pi}$$

Hence,

$$\begin{aligned} z_1 z_2 = (2\cdot 3)e^{i(\pi+\pi)} &= 6e^{2\pi i} \\ &= 6[\cos(2\pi) + i\sin(2\pi)] \\ &= 6[1+0i] \\ &= 6 \checkmark \end{aligned}$$

So we see it's <u>NOT</u> true that

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2),$$

but it <u>IS</u> true that

$$\boxed{\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)},$$

because now we have the "extra freedom" arg gives us over Arg.

Next fact: from polar form $z = re^{i\theta}$, we see

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

Consequence: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i\theta_1} e^{-i\theta_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

and so,

$$\boxed{\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)}$$

Ex: Find $\operatorname{Arg}\left(\frac{-2}{1+\sqrt{3}i}\right)$

$\arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$|1+\sqrt{3}| = \sqrt{1+3} = \sqrt{4} = 2$

Soln: Let $z_1 = -2 = 2e^{i\pi}$ and let $z_2 = 1+\sqrt{3}i$
$= 2e^{i\pi/3}$

Therefore,

$$\begin{aligned}\arg\left(\frac{z_1}{z_2}\right) &= \arg\left(\frac{-2}{1+\sqrt{3}i}\right) \\ &= \arg(2e^{i\pi}) - \arg(2e^{i\frac{\pi}{3}}) \\ &= (\pi + 2n_1\pi) - \left(\frac{\pi}{3} + 2n_2\pi\right);\ n_1, n_2 \in \mathbb{Z} \\ &= \frac{2\pi}{3} + 2n\pi;\ n = n_1 - n_2 \in \mathbb{Z}\end{aligned}$$

We must find n so that we get principal Arg ~ in this case $n=0$ is it and we have

$$\operatorname{Arg}\left(\frac{-2}{1+\sqrt{3}i}\right) = \frac{2\pi}{3}$$

Ex: Find $\operatorname{Arg}\left(\dfrac{-\sqrt{3}-i}{-\sqrt{3}/2+\frac{1}{2}i}\right)$

Soln: In this case, let

$$z_1 = -\sqrt{3}-i = 2e^{-\frac{5\pi}{6}i}$$

and

$$z_2 = -\frac{\sqrt{3}}{2}+\frac{1}{2}i = 1e^{\frac{5\pi}{6}i}$$

Then,

$$\arg\left(\frac{-\sqrt{3}-i}{-\frac{\sqrt{3}}{2}+\frac{1}{2}i}\right) = \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2)$$
$$= \left(-\frac{5\pi}{6}+2n_1\pi\right)-\left(\frac{5\pi}{6}+2n_2\pi\right)$$
$$= -\frac{10\pi}{6}+2n\pi,\quad n\in\mathbb{Z}$$

Here we must pick $n=1$ so we get

$$\operatorname{Arg}\left(\frac{-\sqrt{3}-i}{-\frac{\sqrt{3}}{2}+\frac{1}{2}i}\right) = -\frac{10\pi}{6}+2\pi = -\frac{10\pi}{6}+\frac{12\pi}{6} = \frac{\pi}{3}\ \checkmark$$

Powers notice that

$$(e^{i\theta})^n = e^{in\theta}$$

interesting consequence: DeMoivre's formula

$$(\cos(\theta)+i\sin(\theta))^n = \cos(n\theta)+i\sin(n\theta)$$

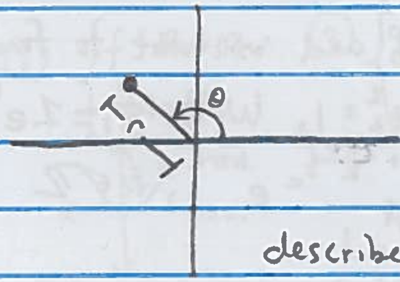
But more practically, can help us compute powers of complex numbers.

Ex: Compute... $(\sqrt{3}+i)^9$

$\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^5$

$(-\sqrt{2}+\sqrt{2}i)^{12}$

Roots of Complex Numbers



From this picture, it is evident that

$r_1 e^{i\theta_1}$ and $r_2 e^{i\theta_2}$

describe the same complex number if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$ for some $k \in \mathbb{Z}$

Def: We say a complex number z_2 is an n^{th} root of a complex number z_1 provided that $z_2^n = z_1$

In polar form, this def says $z_2 = r_2 e^{i\theta_2}$ is an n^{th} root of $z_1 = r_1 e^{i\theta_1}$ if and only if

$$z_2^n = r_2^n \left(e^{i\theta_2}\right)^n = r_2^n e^{in\theta_2} = r_1 e^{i\theta_1},$$

or by top of this page,

$$\begin{cases} r_2^n = r_1 \text{ and} \\ n\theta_2 = \theta_1 + 2k\pi, \; k \in \mathbb{Z} \end{cases}$$

Hence

$$\begin{cases} r_2 = \sqrt[n]{r_1} \leftarrow \text{always non-negative real \#} \\ \theta_2 = \dfrac{\theta_1}{n} + \dfrac{2k\pi}{n}, \; k \in \mathbb{Z} \end{cases}$$

are all of the n^{th} roots of z_1.

Generally speaking: You will find n distinct n^{th} roots of any nonzero complex number!!

Ex: Find all square roots of 1

Soln: Here we have $z_1 = 1$ and we want to find all z_2 so that $z_2^2 = 1$. Write $z_1 = 1e^{0\pi i}$ and so $z_2 = \sqrt{1}\, e^{0 + \frac{2k\pi}{2} i} = e^{k\pi i}$, $k \in \mathbb{Z}$

Taking
$k=0 \rightsquigarrow z_2 = e^{0\pi i} = 1$

$k=1 \rightsquigarrow z_2 = e^{\pi i} = -1$

$k=2 \rightsquigarrow z_2 = e^{2\pi i} = 1$

$k=3 \rightsquigarrow z_2 = e^{3\pi i} = -1$

$\vdots$

Therefore the square roots of 1 are 1 and -1.

Ex: Find all cube roots of 1

Ex: Find cube roots of $(-8i)$

Ex: Find all 5^{th} roots of $2 - 2\sqrt{3}i$ arg is $-\frac{\pi}{3}$

Functions A function is 4 things:

1. a name
2. a domain (set of inputs)
3. a codomain (where outputs live)
4. rule of assignment

We write

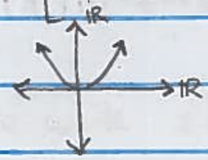
NAME, domain, codomain

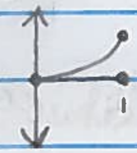
$$\begin{cases} f : A \to B \\ f(x) = \text{blablabla} \end{cases}$$

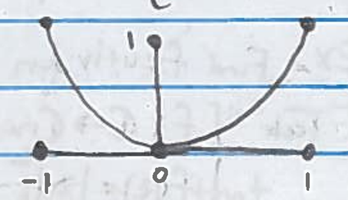
rule of assignment

Ex: Draw each function:

① $\begin{cases} f: \mathbb{R} \to \mathbb{R} \\ f(x) = x^2 \end{cases}$ ② $\begin{cases} g: [0,1] \to \mathbb{R} \\ g(x) = x^2 \end{cases}$ ③ $\begin{cases} h: [-1,1] \to [0,1] \\ h(x) = x^2 \end{cases}$







Convention: if domain is not specified, then we take the largest set as possible for it

fact: if $\begin{cases} f: \mathcal{U} \to \mathbb{C} \\ f(z) = \text{——} \end{cases}$ is a function, then we often will think about its real and imaginary parts and write

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

where $z = x+iy$ and $u, v: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are real-valued functions. We may also write real and imaginary parts of polar form as

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta).$$

Example: Write $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = z^2 \end{cases}$ as $f = u + iv$

Soln: Let $z = x+iy$ and compute $f(z) = (x+iy)^2 = x^2 + 2xyi + i^2y^2$

$$= (x^2 - y^2) + 2xy\,i$$

$u(x,y) = x^2 - y^2$ $v(x,y) = 2xy$

If we had used polar coords,

$$f(z) = (re^{i\theta})^2 = r^2 e^{2\theta i} = \underbrace{r^2\cos(2\theta)}_{u(r,\theta)} + \underbrace{r^2\sin(2\theta)}_{v(r,\theta)} i$$

Ex: Find $f = u + iv$ for

$$\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = |z|^2 \end{cases}$$

Soln:

$$f(z) = |x+iy|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$$

$$= \underbrace{(x^2+y^2)}_{\substack{u(x,y) \\ \| \\ x^2+y^2}} + \underbrace{0}_{v(x,y)=0} i$$

Ex: Find $f = u + iv$ for $\begin{cases} f: \mathbb{C}\setminus\{0\} \to \mathbb{C} \\ f(z) = \frac{1}{z} \end{cases}$

Soln:

$$f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + \left(\frac{-y}{x^2+y^2}\right) i$$

Ex: Write $\begin{cases} f: \mathbb{C} \to \mathbb{C} \\ f(z) = z^2 + 2z + (1+i) \end{cases}$ into $f = u + iv$

Ans: $(x^2 + 2x - y^2 + 1) + (2xy + 2y + 1)i$

in symbols : $\forall \epsilon>0\, \exists \delta>0 \left(|z-z_0|<\delta \rightarrow |f(z)-L|<\epsilon\right)$

Limits

We now give proper meaning to the expression

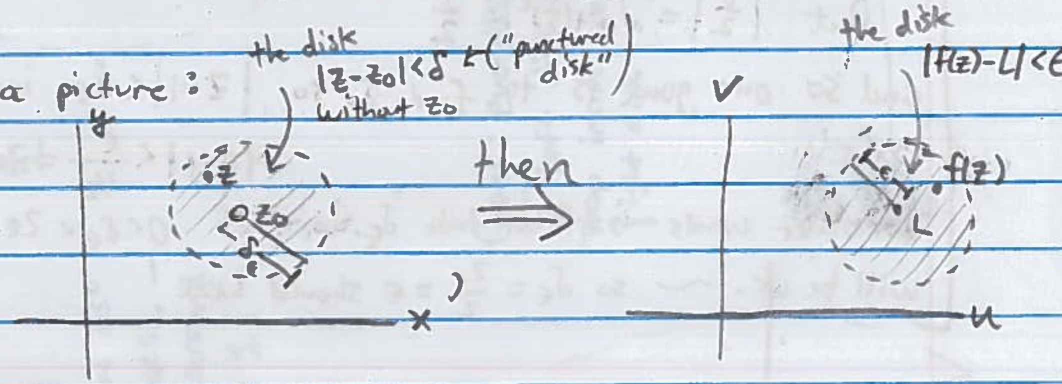
$$\lim_{z \to z_0} f(z) = L$$

Def: We write $\lim_{z \to z_0} f(z) = L$ whenever

FORMAL "for every $\epsilon>0$, there exists $\delta_\epsilon>0$ such that "if $0<|z-z_0|<\delta$, then $|f(z)-L|<\epsilon$""

Draw a picture :

visual if



practical if "z is δ_ϵ-close to z_0", then "$f(z)$ is ϵ-close to L"

Ex: Prove that $\lim_{z \to i} i\frac{z}{2} = \frac{i}{2}$

Soln: Scratch work : let $\epsilon>0$ ~ goal: find δ_ϵ so

scratch work ⤳ do not submit

we can conclude

$$\left|f(z)-\frac{i}{2}\right|<\epsilon$$

(aka)

$$\left|i\frac{z}{2}-\frac{i}{2}\right|<\epsilon$$

from the inequality $|z-1|<\delta_\epsilon$

scratch work
do not submit

So our goal $\left|\frac{iz}{2}-\frac{i}{2}\right|<\epsilon$

$$\Downarrow$$

$$\left|\frac{i}{2}(z-1)\right|<\epsilon$$

$$\left|\frac{i}{2}\right|\,|z-1|<\epsilon$$

$$\Rightarrow |z-1|<\frac{\epsilon}{|i/2|}$$

But $\left|\frac{i}{2}\right|=\sqrt{0^2+(\frac{1}{2})^2}=\frac{1}{2}$

and so our goal is to find δ_ϵ so $|z-1|<\delta_\epsilon$ implies

$$|z-1|<\frac{\epsilon}{1/2}=2\epsilon.$$

In other words $\rightsquigarrow$ if we take δ_ϵ so that $0<\delta_\epsilon<2\epsilon$... we will be ok. $\sim$ so $\delta_\epsilon=\frac{2\epsilon}{2}=\epsilon$ should work!

What I want to see

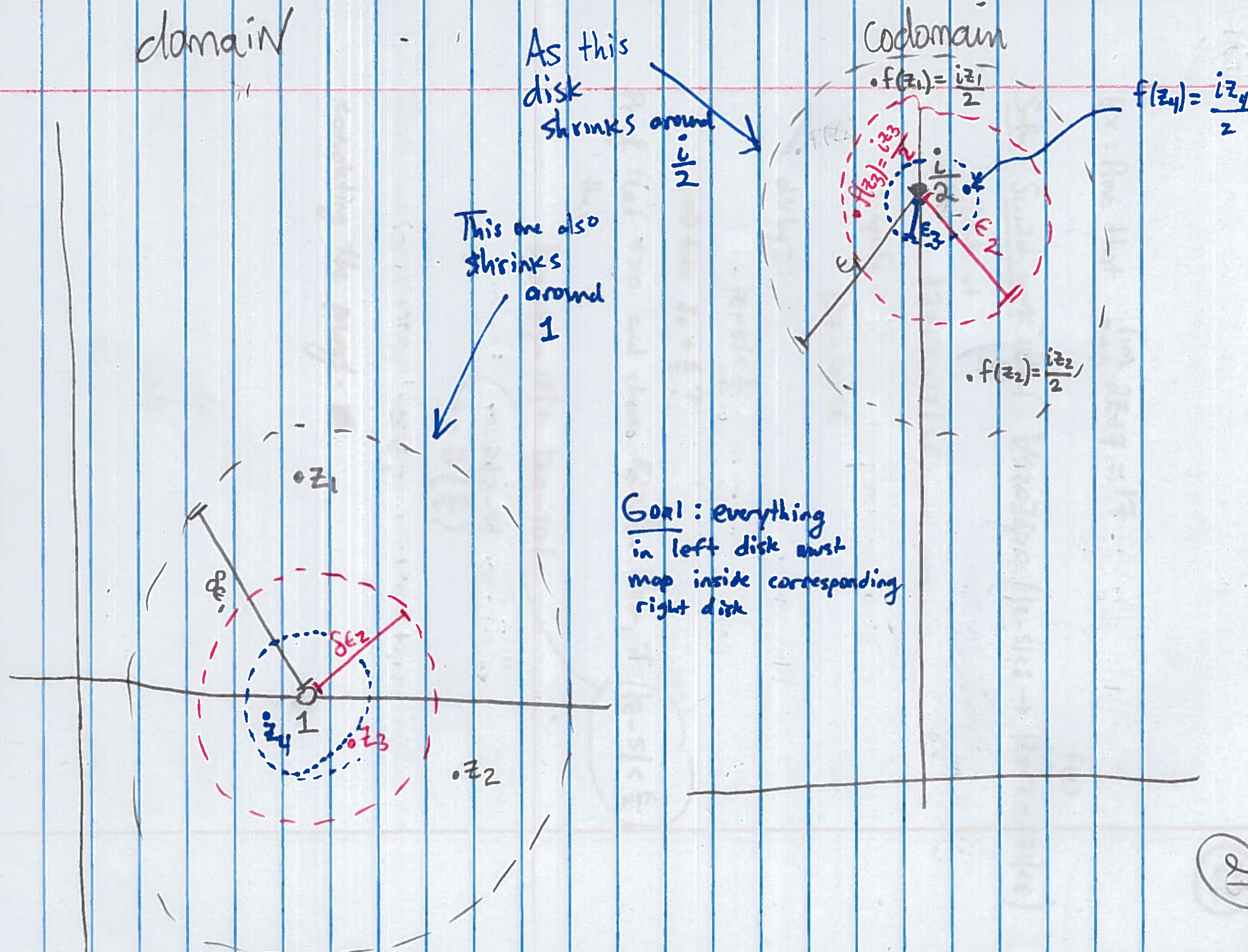
<u>Proof</u>: Let $\epsilon>0$ and pick $\delta_\epsilon=\epsilon$. If $|z-1|<\epsilon$, then

$$\left|f(z)-\frac{i}{2}\right|=\left|\frac{iz}{2}-\frac{i}{2}\right|=\left|\frac{i}{2}\right|\,|z-1|$$

$$=\frac{1}{2}|z-1|$$

$$<\frac{\epsilon}{2}<\epsilon,$$

Completing the proof. ▣



Goal: everything in left disk must map inside corresponding right disk

<u>Ex</u>: Prove that $\lim_{z \to 5} 2z+7 = 17$

<u>Soln</u>: <u>Scratch work</u>: want $\forall \epsilon > 0 \, \exists \delta_\epsilon > 0 \left(|z-5| < \delta \rightarrow |\overbrace{2z+7}^{f(z)} - 17| < \epsilon \right)$

look at

$$|2z+7-17| < \epsilon$$

simplify:

$$|2z-10| < \epsilon$$

div by 2:

$$|z-5| < \frac{\epsilon}{2}$$

$\Longrightarrow$ take $\delta_\epsilon = \frac{\epsilon}{2}$?

<u>Proof</u>: Let $\epsilon > 0$ and choose $\delta_\epsilon = \frac{\epsilon}{2}$. Now, if $|z-5| < \frac{\epsilon}{2}$, then

$$\begin{aligned} |(2z+7) - 17| &= |2z-10| \\ &= 2|z-5| \\ &< 2\left(\frac{\epsilon}{2}\right) \\ &= \epsilon, \end{aligned}$$

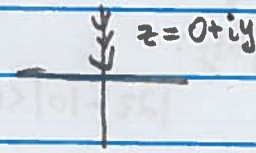
completing the proof. ∎

Ex: Show that $\lim\limits_{z \to 0} \dfrac{z}{\bar{z}}$ does not exist.

Soln: We get different values depending on how z approaches zero:

Case 1: vertical axis

if $z \to 0$ on imag axis

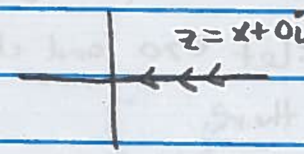


Then we get

$$\lim_{0+iy \to 0} \frac{0+iy}{0-iy} = \lim_{0+iy \to 0} -1 = -1$$

Case 2: horiz axis

if $z \to 0$ on real axis



Then

$$\lim_{x+0i \to 0} \frac{x+0i}{x-0i} = \lim_{x+0i \to 0} 1 = 1$$

Different values on different approach $\Rightarrow$ limit DNE