Complex Numbers + their arithmetic

Complex numbers are of the form $a+bi$ where a and b are real numbers, and i obeys the property $i^2=-1$.

Notation: • $x \in Y$ ~ means x is a member (or element) of the set Y

• $\mathbb{R}$ – set of real numbers

• $\mathbb{C}$ – set of complex numbers

Arithmetic works as expected:

• $(a+bi)+(c+di) = (a+c)+(b+d)i$

• $(a+bi)(c+di) = ac+adi+bci+bdi^2$

$$\stackrel{(i^2=-1)}{=} (ac-bd)+(ad+bc)i$$

Useful trick – to remove complex numbers from denominator, multiply by a "convenient form of 1" as follows:

$$\frac{a+bi}{c+di} = \left(\frac{a+bi}{c+di}\right)\underbrace{\left(\frac{c-di}{c-di}\right)}_{=1} = \frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-d^2i^2}$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

Ex: Write in the form $a+bi$, where $a, b \in \mathbb{R}$.

- $(1-i)(2+3i) = \ldots$

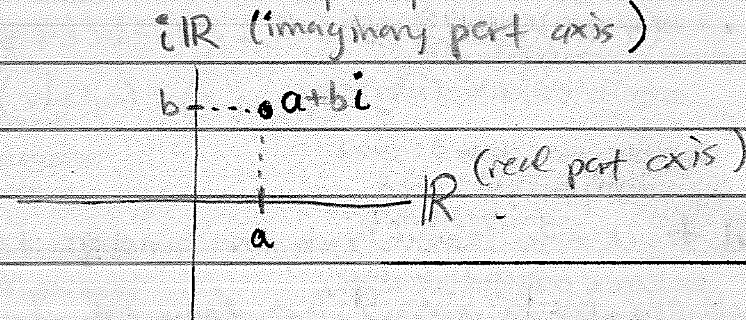
- $\dfrac{2+i}{1-3i} = \ldots$

If a complex number $z \in \mathbb{C}$ is written in form

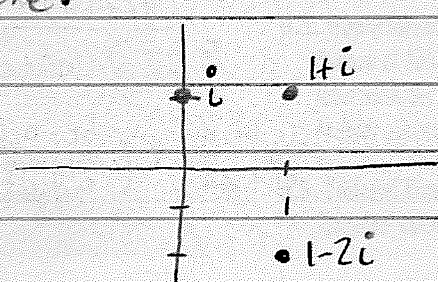
$$z = a + bi$$

we say the real part of z is a (write $\mathrm{Re}(z) = a$) and the imaginary part of z is b (write $\mathrm{Im}(z) = b$).

We often will think of a complex number $z = a+bi$ as the point (a, b) in the complex plane



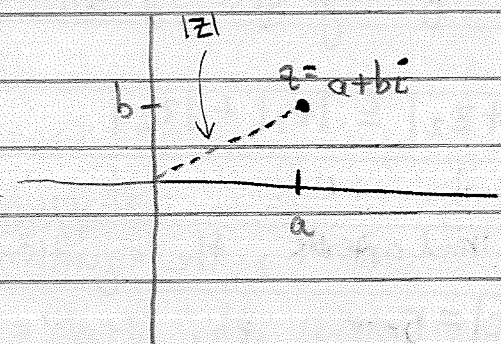
Example: Plot i, $1+i$, and $1-2i$ in the complex plane.



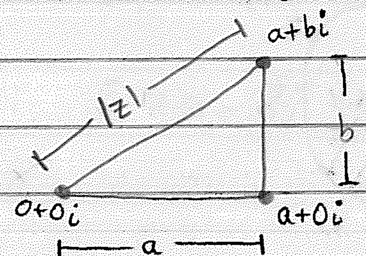
Modulus

Given any $z=a+bi$, the modulus of z is the distance between z and the origin in the complex plane.

We write $|z|$ to denote the modulus of z.



notice we have a triangle:



By pythagorean theorem,

$$|z|^2 = a^2+b^2,$$

so $$|z| = {}^{+}\sqrt{a^2+b^2}$$

Ex: Compute

$$|3+i| = \_\_\_$$

$$\left|\frac{1}{2+i}\right| = \_\_\_$$

Properties of Modulus

(1.) Triangle inequality: for any $z_1, z_2 \in \mathbb{C}$,

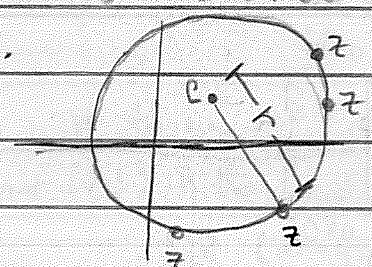
$$|z_1+z_2| \leq |z_1| + |z_2|$$

(2.) for $c \in \mathbb{C}$ and $r \in \mathbb{R}$, the equation (with variable z)

$$|z-c|=r$$

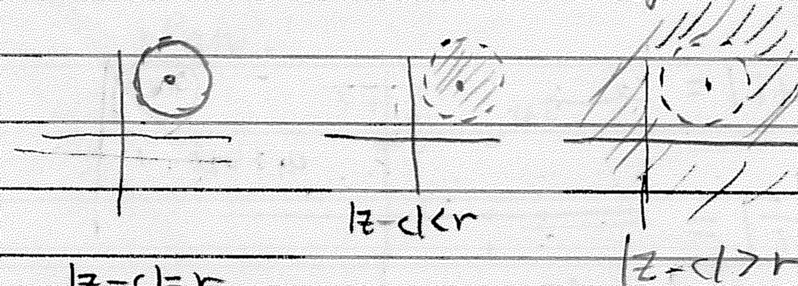
describes the circle of radius r centered at c,

i.e.



but not boundary

(3.) The inequality $|z-c|<r$ denotes the interior of the circle $|z-c|=r$. The inequality $|z-c|>r$ denotes all points outside of the circle $|z-c|=r$. If $<$ is replaced by $\leq$ or $>$ replaced by $\geq$, then the sets DO include boundary circle.



Ex: Draw $|z|<1$.

Draw $|z+(2+i)|>2$

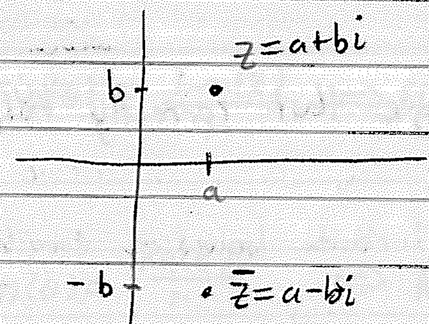
Complex Conjugate

Given $z=a+bi$, the complex conjugate of z, written $\bar{z}$, is defined by

$$\bar{z}=a-ib$$

Properties

- Geometrically behaves like a reflection across the horizontal axis



- if $z_1, z_2 \in \mathbb{C}$, then $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

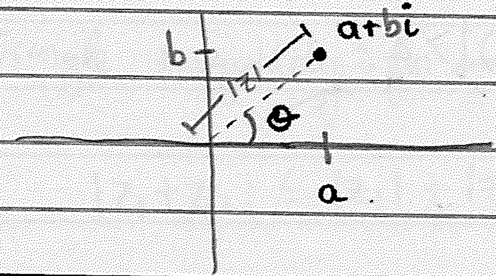
  Proof: If $z_1=a_1+b_1i$, $z_2=a_2+b_2i$, then

  $\overline{z_1+z_2}=\dots$

- if $z\in\mathbb{C}$, then $\bar{\bar{z}}=z$

Argument

Given a $z \in \mathbb{C}$ with $z = a + bi$ we know how to plot it:



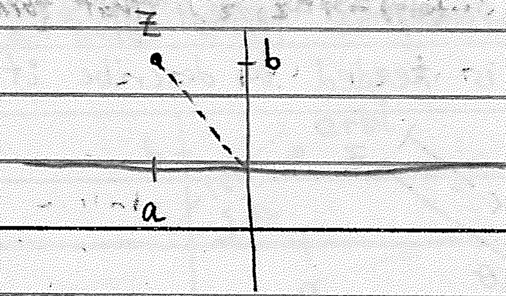
The angle θ in the picture, with $-\pi < \theta \leq \pi$, is called the principal argument of z, written as $\mathrm{Arg}(z)$.

Of course any number of full positive or negative rotations from $\mathrm{Arg}(z)$ is still "an" angle that can help describe z. In that direction we use the notation

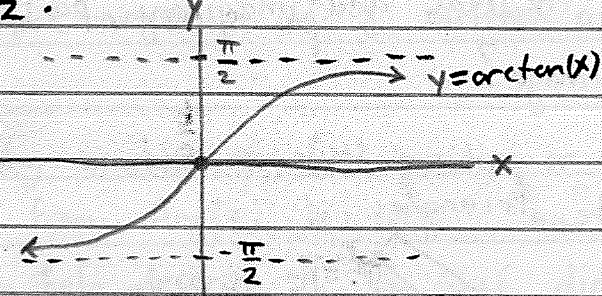
$$\arg(z) = \mathrm{Arg}(z) + 2n\pi \quad , \; n \in \{0, \pm 1, \pm 2, \dots\}$$

to refer to any angle that correctly helps us identify z.

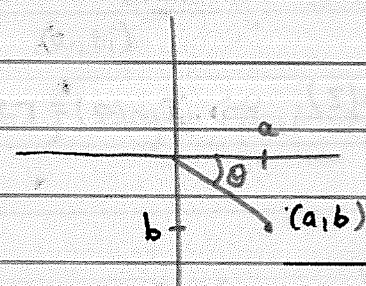
Finding radius and principal Arg from real + imaginary parts

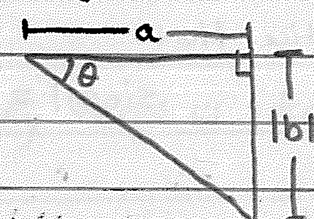


Recall the arctan function (i.e. inverse tangent) takes any real number and outputs a number strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.



Also recall that the angle of any point (a,b) in QI or QIV may be found in the following way:



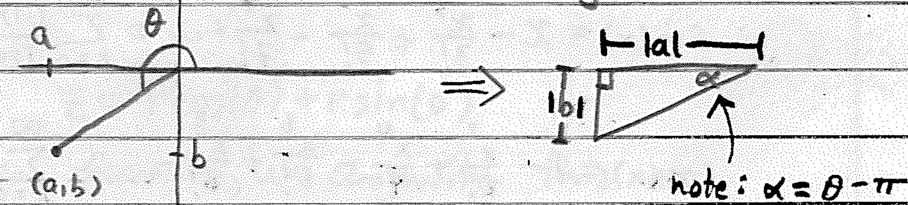


$$\tan(\theta) = \frac{b}{a}$$

$$\Downarrow$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

However, since range(arctan) $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that formula does not work for a point in Q II or Q III. So we do the following:



$$\tan(\alpha) = \frac{b}{a}$$

$$\alpha = \arctan\left(\frac{b}{a}\right)$$

$$\Downarrow$$

$$\theta = \pi + \arctan\left(\frac{b}{a}\right)$$

In summary, given $z = a+bi$,

$$r = \sqrt{a^2+b^2}$$

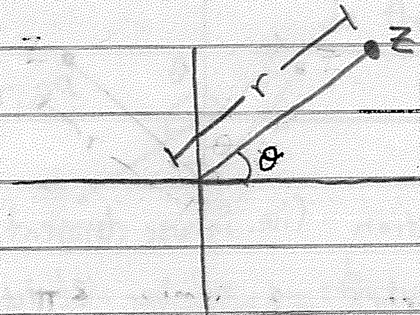
$$\arg(z) = \begin{cases} \arctan\left(\frac{b}{a}\right) + 2n\pi, & a+bi \text{ in QI or QIV} \\ \pi + \arctan\left(\frac{b}{a}\right) + 2n\pi, & a+bi \text{ in QII or QIII} \end{cases}$$

<u>Ex:</u> Find $|z|$, Arg(z), and three other values of arg(z) for...

- $z = 1+2i$
- $z = 3i$
- $z = -1-i$

Finding real and imaginary parts from modulus and principal Arg

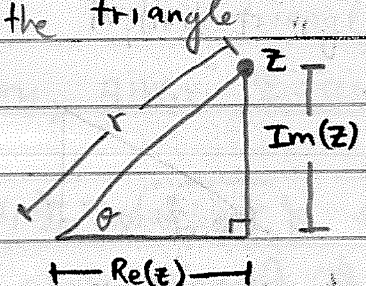
The modulus and the (principal) argument of a complex number are all that is needed to describe it.



What are the real and imaginary parts of z in the above image?

Consider the triangle

$\frac{\pi}{2} + (\pi - \theta) + \theta$



Notice that

$$\sin(\theta) = \frac{Im(z)}{r} \Rightarrow Im(z) = r\sin(\theta)$$

and

$$\cos(\theta) = \frac{Re(z)}{r} \Rightarrow Re(z) = r\cos(\theta)$$

Therefore this z is

$$z = r\cos(\theta) + i r\sin(\theta)$$

Euler's formula

Recall the following Taylor series from calc 2:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Also powers of i obey

$$\begin{tikzcd}
& 1 = i^4 = i^8 \cdots \arrow[dr, bend left, "\text{mult by } i"] & \\
\cdots i^{11} = i^7 = i^3 = -i \arrow[ur, bend left] & & i = i^5 = i^9 \cdots \arrow[dl, bend left, "\text{mult by } i"] \\
& -1 = i^2 = i^6 = i^{10} \cdots \arrow[ul, bend left] &
\end{tikzcd}$$

So we observe something interesting:

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) i$$

$$= \cos(x) + i\sin(x)$$

That formula is called Euler's formula.

Notice that on p. 9, we arrived at

$$z = r\cos(\theta) + r\sin(\theta)$$

Using Euler's formula, we can write this as

$$\begin{aligned} z &= r\cos(\theta) + r\sin(\theta) \\ &= r\left[\cos(\theta) + i\sin(\theta)\right] \\ &= re^{i\theta} \end{aligned}$$

The form $z = re^{i\theta}$ is called the polar form of z.

Consequently,

$$r = |z|$$

and

$$\theta = \arg(z)$$