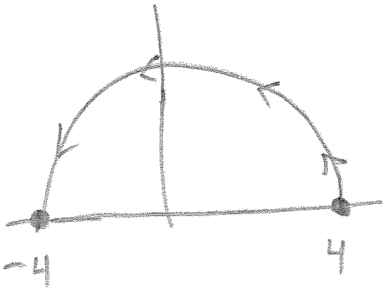
Homework 7 MATH 1199 Fall 2019

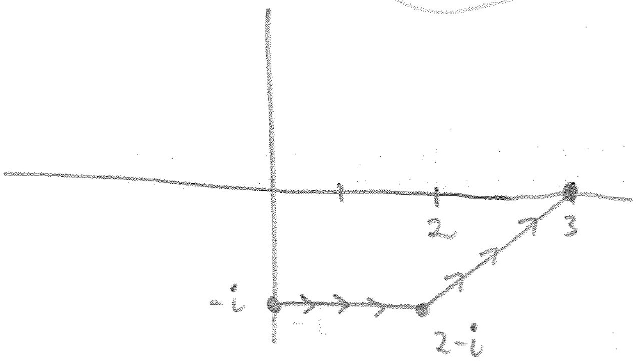
① Draw...

a) $\begin{cases} z(t)=4e^{it} \\ 0\le t\le \pi \end{cases}$

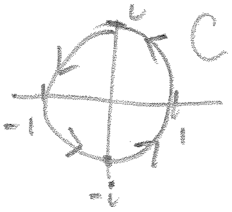


(b) $z(t)=\begin{cases} t-i, & 0\le t\le 2 \\ t+(t-3)i, & 2\le t\le 3 \end{cases}$

$z(0)=-i$, $\boxed{z(2)=2-i = 2+(2-3)i}$, $z(3)=3+(3-3)i=3$



② a) $\int_C \overset{f(z)}{\frac{1}{z}}\,dz$, $C: \begin{cases} z(t)=e^{it} \\ 0\le t\le 2\pi \end{cases}$



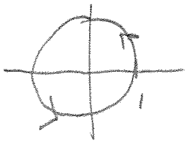
Soln: $f(z(t))=\frac{1}{e^{it}}$
$z'(t)=ie^{it}$

$$\Rightarrow \int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{it}}(ie^{it})\,dt$$

$$= i\int_0^{2\pi} 1\,dt$$

$$= 2\pi i$$

b) $\int_C \overset{f(z)}{\boxed{\frac{1}{z^2}}}\,dz$, $C : \begin{cases} z(t) = e^{it} \\ 0 \le t \le 2\pi \end{cases}$



Soln: $f(z(t)) = \frac{1}{(e^{it})^2} = \frac{1}{e^{2it}}$

$z'(t) = ie^{it}$

So, calculate

$$\int_C \frac{1}{z^2}\,dz = \int_0^{2\pi} \frac{1}{e^{2it}}\left(ie^{it}\right)dt$$

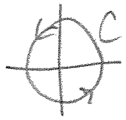
$$= i\int_0^{2\pi} e^{-it}\,dt$$

$$= \frac{i}{-i}\,e^{-it}\Big|_0^{2\pi}$$

$$=(-1)\Big[\underbrace{e^{-2\pi i}}_{=1} - \underbrace{e^{0}}_{=1}\Big]$$

$$=0$$

d) $\int_C z^2 + 2z\,dz$,



Soln: $f(z(t)) = e^{2it} + 2e^{it}$

$z'(t) = ie^{it}$

So calculate

$$\int_C z^2 + 2z\,dz = \int_0^{2\pi} \left(e^{2it} + 2e^{it}\right)ie^{it}\,dt$$

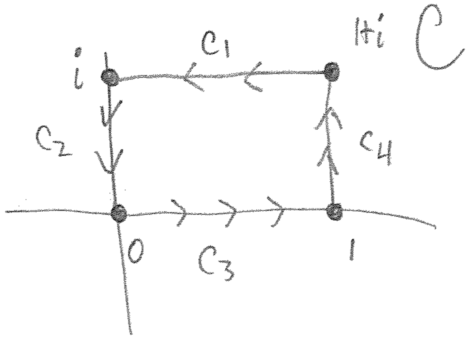
$$= i\int_0^{2\pi} e^{3it} + 2e^{2it}\,dt$$

$$= i\left[\frac{1}{3i}e^{it} + \frac{2}{2}e^{2it}\right|_0^{2\pi}$$

$$= i\left[\left(\frac{1}{3i}\underbrace{e^{2\pi i}}_{=1} + \underbrace{e^{4\pi i}}_{=1}\right) - \left(\frac{1}{3i}\underbrace{e^{0}}_{=1} + \underbrace{e^{0}}_{=1}\right)\right]$$

$$= 0$$

③



③

C_1

Parametrize: $\begin{cases} z_1(t) = ti + (1-t)(1+i) \\ \quad = \cancel{ti} + 1 + i - t - \cancel{ti} \\ \quad = (1-t) + i \\ 0 \leq t \leq 1 \end{cases}$

$z_1'(t) = -1$

$f(z_1(t)) = \pi \exp\left(\pi\left(\overline{(1-t)+i}\right)\right)$

$= \pi \exp(\pi(1-t-i))$

$= \pi e^{(1-i)\pi} e^{-\pi t}$

$\pi \int_{C_1} \exp(\pi \bar{z})\, dz = \pi e^{(1-i)\pi} \int_0^1 e^{-\pi t}\, dt$

$= \cancel{\pi} e^{(1-i)\pi} \left[-\frac{1}{\cancel{\pi}} e^{-\pi t} \Big|_0^1 \right]$

$= -e^{(1-i)\pi} [e^{-\pi} - 1]$

$= -e^{\pi} \underbrace{e^{-\pi i}}_{=-1} [e^{-\pi} - 1]$

$= e^{\pi} [e^{-\pi} - 1]$

$= 1 - e^{\pi}$

C_2

Parametrize: $\begin{cases} z_2(t) = t(0) + (1-t)i \\ \quad = (1-t)i \\ 0 \leq t \leq 1 \end{cases}$

$z_2'(t) = -i$

$f(z_2(t)) = \pi \exp\left(\pi\left(\overline{(1-t)i}\right)\right)$

$= \pi \exp(-\pi(1-t)i)$

$= \pi \underbrace{e^{-\pi i}}_{=-1} e^{\pi t i}$

$= -\pi e^{\pi t i}$

$\pi \int_{C_2} \exp(\pi \bar{z})\, dz = -\pi \int_0^1 e^{(\pi i)t} (-i)\, dt$

$= \pi \cancel{i} \left[\frac{1}{\cancel{\pi i}} e^{\pi i t} \Big|_0^1 \right]$

$= e^{\pi i} - e^{0}$

$= -1 - 1$

$= -2$

C_3

Parametrize: $\begin{cases} z_3(t) = t(1)+(1-t)(0) \\ \qquad = t \\ 0 \le t \le 1 \end{cases}$

$z_3'(t) = 1$

$f(z_3(t)) = \pi \exp(\pi \cdot \bar{t}) = \pi e^{\pi t}$

$$\pi \int_{C_3} \exp(\pi \bar{z})\,dz = \pi \int_0^1 e^{\pi t}(1)\,dt$$

$$= \cancel{\pi}\left[\frac{1}{\cancel{\pi}} e^{\pi t}\right|_0^1$$

$$= e^{\pi} - e^{0}$$

$$= e^{\pi} - 1$$

C_4

Parametrize: $\begin{cases} z_4(t) = t(1+i)+(1-t)(1) \\ \qquad = \cancel{t}+ti+1-\cancel{t} \\ \qquad = 1+ti \\ 0 \le t \le 1 \end{cases}$

$z_4'(t) = i$

$$f(z_4(t)) = \pi \exp(\pi(\overline{1+ti}))$$

$$= \pi \exp(\pi(1-ti))$$

$$= \pi e^{\pi} e^{-\pi t i}$$

$$\pi \int_{C_4} \exp(\pi \bar{z})\,dz = \pi \int_0^1 e^{\pi} e^{-\pi t i}(i)\,dt$$

$$= \pi e^{\pi} i \int_0^1 e^{-\pi t i}\,dt$$

$$= \cancel{\pi i} e^{\pi}\left[\frac{1}{\cancel{-\pi i}} e^{-\pi t i}\right|_0^1$$

$$= -e^{\pi}\left[\underbrace{e^{-\pi i}}_{=-1} - e^{0}\right]$$

$$= -e^{\pi}[-2]$$

$$= 2e^{\pi}$$

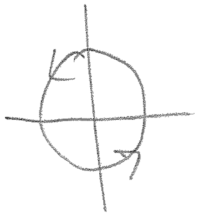
Therefore, finally,

$$\int_C \pi \exp(\pi \bar{z})\,dz = \left[\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right] \pi \exp(\pi \bar{z})\,dz$$

$$= (\cancel{1}-\cancel{e^{\pi}}) + (-2) + (\cancel{e^{\pi}}-\cancel{1}) + 2e^{\pi}$$

$$= -2 + 2e^{\pi}$$

④



$$\begin{cases} z(t) = e^{it} \\ 0 \le t \le 2\pi \end{cases}$$

$f(z) = z^n \longrightarrow f(z(t)) = (e^{it})^n = e^{nit}$; $n \neq -1$

$z'(t) = ie^{it}$

So calculate

~~$\underline{n=0}$~~

~~$\int_C z^n dz = \int_C 1 \, dz$~~

~~$= \int_0^{2\pi} (1)(ie^{it}) dt$~~

~~$=$~~

$$\int_C z^n dz = \int_0^{2\pi} (e^{nit})(ie^{it}) dt$$

$$= i \int_0^{2\pi} e^{(n+1)it} dt$$

$$= i \left[\frac{1}{(n+1)i} e^{(n+1)it} \right|_0^{2\pi}$$

$n \neq -1$ necessary to write this!!

$$= \frac{1}{n+1} \left[\underbrace{e^{(n+1)\cdot 2\pi i}}_{=1} - \underbrace{e^0}_{=1} \right]$$

$$= 0$$