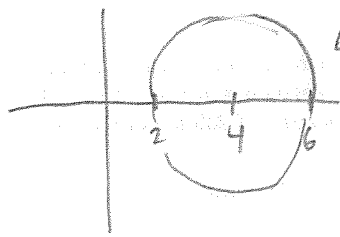
Homework 6 Math 1199 Fall 2019

1) Where does 

circle $|z-4|=2$ map to under $f(z)=\frac{1}{z}$?

$$w \in f[\text{circle}] \longleftrightarrow \frac{1}{w} \in \text{circle}$$

$$\longleftrightarrow \left|\frac{1}{w}-4\right| = 2$$

$\downarrow$ multi. by $|w|$

$$|1-4w| = 2|w|$$

$\downarrow$ square

$$|1-4w|^2 = 4|w|^2$$

$\downarrow$ using $|z|^2 = z\bar{z}$

$$(1-4w)(1-4\bar{w}) = 4w\bar{w}$$

$\frac{1}{3} = \frac{4}{12}$

$$\downarrow$$

$$1-4\bar{w}-4w+16w\bar{w} = 4w\bar{w}$$

$$\downarrow$$

$$12w\bar{w}-4\bar{w}-4w = -1$$

$$w\bar{w} - \frac{1}{3}\bar{w} - \frac{1}{3}w = -\frac{1}{12}$$

$\downarrow$ add $\left(\frac{1}{3}\right)^2$ ("algebra trick")

$$w\bar{w} - \frac{1}{3}\bar{w} - \frac{1}{3}w + \left(\frac{1}{3}\right)^2 = -\frac{1}{12} + \left(\frac{1}{3}\right)^2$$

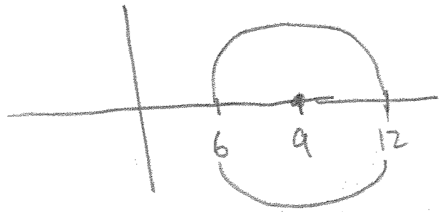
$$\bar{w}\left(w-\frac{1}{3}\right) - \frac{1}{3}\left(w-\frac{1}{3}\right) = \frac{1}{36}$$

$$\left(\bar{w}-\frac{1}{3}\right)\left(w-\frac{1}{3}\right) = \frac{1}{36}$$

$$\left|w-\frac{1}{3}\right|^2 = \frac{1}{36}$$

$$\left|w-\frac{1}{3}\right| = \frac{1}{6}$$

So f maps the circle to the circle of radius $\frac{1}{6}$ centered at $\frac{1}{3}$.

(2) Where does 

map to under $f(z) = \frac{1}{z}$? (2)

$$w \in f[\text{circle}] \longleftrightarrow \frac{1}{w} \in \text{circle}$$

$$\longleftrightarrow \left|\frac{1}{w} - 9\right| = 3$$

$$\longleftrightarrow |1 - 9w| = 3|w|$$

$$|1-9w|^2 = 9|w|^2$$

$$(1-9w)(1-9\bar{w}) = 9w\bar{w}$$

$$1 - 9\bar{w} + 9w + 81w\bar{w} = 9w\bar{w}$$

$$72w\bar{w} - 9\bar{w} - 9w = -1$$

$$w\bar{w} - \frac{9}{72}\bar{w} - \frac{9}{72}w = -\frac{1}{72}$$

$\frac{9}{72} = \frac{1}{8}$

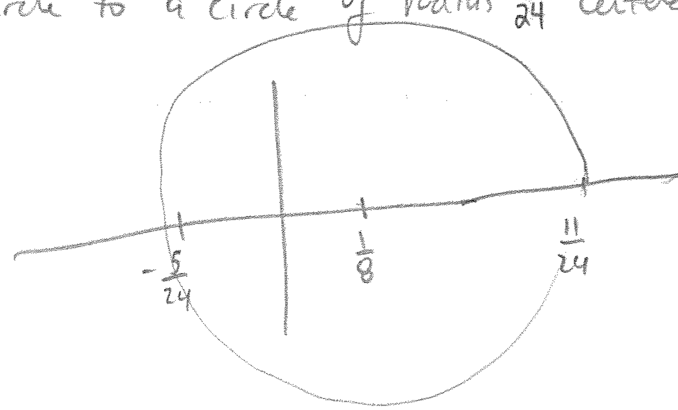
$$w\bar{w} - \frac{1}{8}\bar{w} - \frac{1}{8}w + \left(\frac{1}{8}\right)^2 = -\frac{1}{72} + \left(\frac{1}{8}\right)^2$$

$$\left(w - \frac{1}{8}\right)\left(\bar{w} - \frac{1}{8}\right) = \frac{1}{576}$$

$$\left|w - \frac{1}{8}\right|^2 = \frac{1}{576}$$

$$\left|w - \frac{1}{8}\right| = \frac{1}{24}$$

So f maps the circle to a circle of radius $\frac{1}{24}$ centered at $\frac{1}{8}$.



(3) a) $6 \mapsto 0$

$-i \mapsto 1$

$64-13i \mapsto \infty$

Soln: Here use the formula from p. 51 of the notes using $z_1 = 6$, $z_2 = -i$, and $z_3 = 64-13i$ to get

$$f(z) = \left(\frac{z-6}{z-(64-13i)}\right)\left(\frac{-i-(64-13i)}{-i-6}\right)$$

b) $-1+i \mapsto 1+i$

$\infty \mapsto 11$

$7+3i \mapsto -99i$

First use formula from p. 51 to get a Möbius transformation f that takes

$$\begin{aligned} -1+i &\mapsto 0 \\ \infty &\mapsto 1 \\ 7+3i &\mapsto \infty \end{aligned} \quad \xrightarrow[z_3 = 7+3i]{z_1=-1+i,\ z_2=\infty} \quad f(z) = \left(\frac{z-(-1+i)}{z-(7+3i)}\right)\cancel{\left(\frac{\infty-(7+3i)}{\infty-(-1+i)}\right)}$$

cancels

$$= \frac{z-(-1+i)}{z-(7+3i)}$$

Second, use same formula to get a M.T. g that takes

$$\begin{aligned} 1+i &\mapsto 0 \\ 11 &\mapsto 1 \\ -99i &\mapsto \infty \end{aligned} \quad \xrightarrow[z_3 = -99i]{z_1=1+i,\ z_2=11} \quad g(z) = \left(\frac{z-(1+i)}{z-(-99i)}\right)\left(\frac{11-(-99i)}{11-(1+i)}\right)$$

$$= \left(\frac{z-(1+i)}{z+99i}\right)\left(\frac{11+99i}{10-i}\right)$$

Now find inverse of $g(z)$:

$$w = \left(\frac{z-(1+i)}{z+99i}\right)\left(\frac{11+99i}{10-i}\right) \implies (10-i)(z+99i)w = (z-(1+i))(11+99i)$$

$$99w - iwz + 990iw + 10wz = (11+99i)z + (88-110i)$$

$$\implies z\left[-iw+10w-(11+99i)\right] = 88-110i-99w-990iw$$

$$\implies z = \frac{(-99-990i)w+(88-110i)}{(10-i)w-(11+99i)} \implies g^{-1}(z) = \frac{(-99-990i)z+(88-110i)}{(10-i)z-(11+99i)}$$

Now the desired map is

$$h(z) = g^{-1}(f(z)) = g^{-1}\left(\frac{z-(-1+i)}{z-(7+3i)}\right)$$

$$= \frac{(-99-990i)\left(\frac{z-(-1+i)}{z-(7+3i)}\right) + (88-110i)}{(10-i)\left(\frac{z-(-1+i)}{z-(7+3i)}\right) - (11+99i)}$$

#4) zeros $\longrightarrow$ black $\longrightarrow$ $2i$ and -1

"blow up" $\longrightarrow$ white $\longrightarrow$ $0.5i$ and 3

#5) branch cut : $(-\infty, 0)$

zeros: $\pi, 2\pi, 3\pi, \ldots$ $\longrightarrow$ turns out that $J_{1/2}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(z)}{\sqrt{z}}$!

blow up: none

#6) branch cut: $(-i, i)$ on vertical axis

pos values: ~~at~~ positive real inputs only

negative : negative real inputs only