Homework 5 MATH 1199 Fall 2019

1 a) $2^{1-i} = e^{(1-i)\log(2)} = e^{(1-i)[\ln|2| + i\arg(2)]}$

$= e^{(1-i)[\ln(2) + 2\pi n i]}$

$= e^{\ln(2) + 2\pi n i - i\ln(2) + 2n\pi}$

$= \underbrace{e^{\ln(2)+2n\pi}}_{= e^{\ln(2)} e^{2n\pi}} e^{(2n\pi - \ln(2))i}$

$= 2e^{2n\pi}\cos(2n\pi - \ln(2)) + 2e^{2n\pi}\sin(2n\pi - \ln(2))i$

b) $(P.V.)\ (1+i)^{2i} = e^{2i\,\mathrm{Log}(1+i)} = e^{2i[\ln|1+i| + i\,\mathrm{Arg}(1+i)]}$

$|1+i| = \sqrt{1+1} = \sqrt{2}$

$= e^{2i\ln(\sqrt{2}) + 2i^2\frac{\pi}{4}}$

$= e^{-\frac{\pi}{2}} e^{i\ln(2)}$

$\ln(\sqrt{2}) = \ln(2^{1/2}) = \frac{1}{2}\ln(2)$

$= e^{-\frac{\pi}{2}}\cos(\ln(2)) + e^{-\frac{\pi}{2}}\sin(\ln(2))i$

c) $1^i = e^{i\log(1)} = e^{i[\underbrace{\ln|1|}_{=0} + 2\pi n i]} = e^{-2\pi n}$

d) $(P.V.)\ (-2)^{-i} = e^{-i\,\mathrm{Log}(-2)} = e^{-i[\ln(|-2|) + i\,\mathrm{Arg}(-2)]}$

$= e^{-i[\ln(2) + \pi i]}$

$= e^{-\pi} e^{-\ln(2)i}$

$= e^{-\pi}\cos(\ln(2)) + e^{-\pi}\sin(\ln(2))i$

$(-i)^2 = i^2 = -1$

e) $\arcsin(-i) = \frac{1}{i}\log\left(i(-i)+\sqrt{1-(-i)^2}\right)$

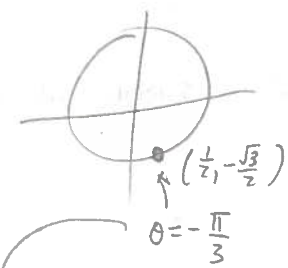
$= \frac{1}{i}\log\left(1+\sqrt{2}\right)$

$= \frac{1}{i}\left[\ln|1+\sqrt{2}| + i\,2\pi n i\right]$

$= \frac{1}{i}\ln\left(1+\sqrt{2}\right) + 2\pi n$

$1+\sqrt{2} \in \mathbb{R}^+$
so $\mathrm{Arg}(1+\sqrt{2}) = 0$
so $\arg(1+\sqrt{2}) = 2n\pi$

f) $\mathrm{Arccos}\left(\frac{1}{2}\right) = \frac{1}{i}\mathrm{Log}\left(\frac{1}{2}+\sqrt{\left(\frac{1}{2}\right)^2-1}\right)$



$= \frac{1}{i}\mathrm{Log}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= \frac{1}{i}\left[\underbrace{\ln \underbrace{\left|\frac{1}{2} - \frac{\sqrt{3}}{2}i\right|}_{=1}}_{=\ln(1)=0} + i\,\mathrm{Arg}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right]$

$= \frac{1}{i}\left[0 + i\frac{\pi}{3}\right] = \frac{\pi}{3}$

2) a) $\frac{d}{dz}\cos(z) = \frac{d}{dz}\,\frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left[ie^{iz} - ie^{-iz}\right]$

$= \frac{i}{2}\left[e^{iz} - e^{-iz}\right]$

$= -\left(\frac{e^{iz}-e^{-iz}}{2i}\right)$

multiply by convenient form of 1 ~ $\frac{i}{i}$

$= -\sin(z)$

b) $\sin(2z) \overset{def}{=} \dfrac{e^{2zi} - e^{-2zi}}{2i}$

Match!!

$$2\sin(z)\cos(z) \overset{def}{=} \cancel{2}\left(\frac{e^{iz} - e^{-iz}}{\cancel{2}i}\right)\left(\frac{e^{iz} + e^{-iz}}{2}\right)$$

$$= \frac{1}{2i}\left[(e^{iz})^2 + \underbrace{\cancel{(e^{iz})(e^{-iz})} - \cancel{(e^{-iz})(e^{iz})}}_{=0} - (e^{-iz})^2\right]$$

$$= \frac{e^{2zi} - e^{-2zi}}{2i}$$

c) $\dfrac{d}{dz}\sinh(z) = \dfrac{d}{dz}\left(\dfrac{e^{z} - e^{-z}}{2}\right) = \dfrac{1}{2}\left[e^{z} - (-e^{-z})\right]$

$$= \frac{e^{z} + e^{-z}}{2}$$

$$\overset{def}{=} \cosh(z)$$

d) $\cosh(iz) \overset{def}{=} \dfrac{e^{iz} + e^{-iz}}{2} \overset{def}{=} \cos(z)$

3) a) $\tan(z) = \dfrac{\sin(z)}{\cos(z)} = \dfrac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = \left(\dfrac{e^{iz} - e^{-iz}}{\cancel{2}i}\right)\left(\dfrac{\cancel{2}}{e^{iz} + e^{-iz}}\right) = \dfrac{1}{i}\left(\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)$

3b) $\tan(w) = z$

$\downarrow$

$$\frac{1}{i}\left(\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}\right) = z$$

$$e^{iw} - e^{-iw} = iz\left(e^{iw} + e^{-iw}\right)$$

$\downarrow$ $v = e^{iw} \Rightarrow \frac{1}{v} = e^{-iw}$

$$v - \frac{1}{v} = izv + \frac{iz}{v} \xrightarrow{\text{mult by } v} v^2 - 1 = izv^2 + iz$$

$\downarrow$

$$(1-iz)v^2 = 1+iz$$

$$v^2 = \frac{1+iz}{1-iz}$$

$$e^{iw} \overset{\text{def}}{=} v = \pm\sqrt{\frac{1+iz}{1-iz}}$$

$\downarrow$ take log

$$iw = \log\left(\pm\sqrt{\frac{1+iz}{1-iz}}\right)$$

$\downarrow$

$$w = \frac{1}{i}\log\left(\pm\sqrt{\frac{1+iz}{1-iz}}\right)$$